Lecture 6: Quasi-categories via inner horns

Quasi-categories are simplicial sets in which one can paste any "well-directed" collection of simplices.

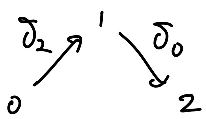
Def: Recall that the k-th horn Λ^n_k of Δ^n is the union of σ_i for $i \neq k$.

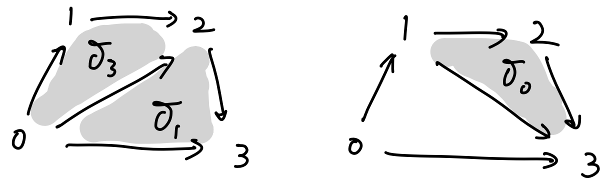
face opposite to vertex i.

We say this horn is **inner** if $0 < k < n$.

e.g.

Λ^2_1	Λ^3_2





Def: A **quasi-category** is a simplicial set X in which any inner horn can be filled to a simplex: for any $0<k<n$,

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\forall"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

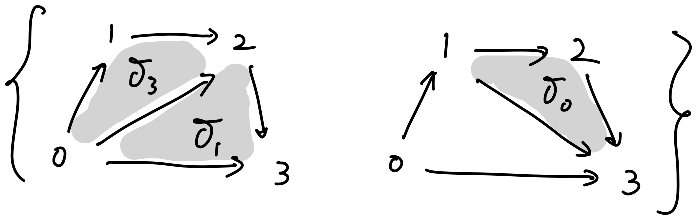
Recall that Kan complexes were defined similarly but with $0 \leq k \leq n$, $\rightsquigarrow$ any Kan complex is a quasi-category

and we interpreted horn filling as pasting of simplices.

σ_k = result of pasting

interior = homotopy witnessing the pasting

Since quasi-categories "are" space-enriched categories, we should be able to paste things within each hom.

e.g. $\Lambda^3_2 =$  $\rightsquigarrow$

$$\begin{tikzcd} \{0,3\} \arrow[d, dash, "\sigma_1"'] & \{0,1,3\} \arrow[d, dash, "\sigma_0"] \\ \{0,2,3\} \arrow[r, dash, "\sigma_3"'] & \{0,1,2,3\} \end{tikzcd}$$

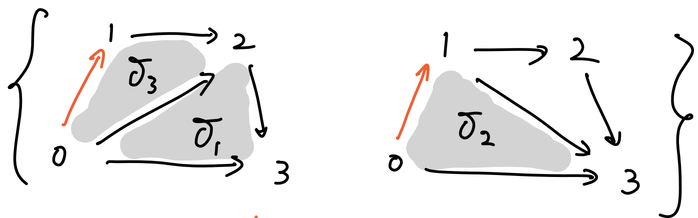
in $\hom_{\mathfrak{C}[3]}(0,3)$.

In general, filling inner $\Lambda^n_k \xrightarrow{f} X$ corresponds to filling an open box in "$\hom_X(f(0), f(n))$".

(More precisely, the action of $\mathfrak{C}(\Lambda^n_k \hookrightarrow \Delta^n)$ on $\hom(0,n)$ is the inclusion of an open box.

This is why $N_{hc}(\mathcal{K})$ is a quasi-category for Kan-enriched $\mathcal{K}$.)

But if $k=0$, we are given things in $\hom(f(0), f(n))$, and we are trying to paste them into something in $\hom(f(1), f(n))$, which shouldn't be possible; similarly for $k=n$.

$\Lambda^3_0 =$ 

points the wrong way.

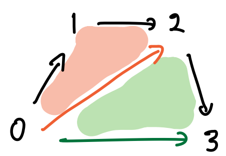
We'll sketch the proof of: Ξ-quasi-categories $=$ Λ-quasi-categories

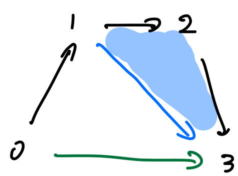
Ξ-qcats $\supset$ Λ-qcats:

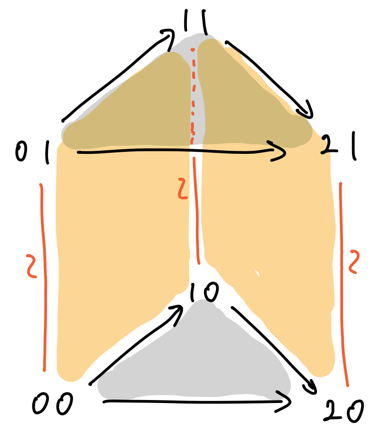
This amounts to showing that any lifting problem
$$\begin{tikzcd} \bullet \arrow[r, "\forall"] \arrow[d, "i"'] & X \\ \bullet \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$
with

$i \in \{\Xi^n \hookrightarrow \Delta^n \mid n \geq 2\} \cup \{(\partial\Delta^n \hookrightarrow \Delta^n) \hat{\times} (\Delta^0 \hookrightarrow J) \mid n \geq 0\}$ can be solved by filling inner horns.

e.g. • When $i : \Xi^3 \hookrightarrow \Delta^3$, we can complete the spine to Λ^3_2 by filling smaller inner horns (Λ^2_1)







• When $i : \Xi^2 \times J \underset{\Xi^2 \times \partial J}{\amalg} \Delta^2 \times \partial J \longrightarrow \Delta^2 \times J$, we are given this & we wish to complete it to a prism. We can obtain the missing simplices by pasting things in $\hom(00, 21)$ & $\hom(01, 20)$.

(Since the vertical edges are invertible, nothing points the wrong way.)

Ξ-qcats $\subset$ Λ-qcats:

We show by induction on n that each inner $\Lambda^n_k \hookrightarrow \Delta^n$ is a trivial cofibration in the Joyal model structure.

$\mathcal{C} \cap \mathcal{W}$

(So fibrant objects, i.e. Ξ-quasi-categories, have the RLP w.r.t. $\Lambda^n_k \hookrightarrow \Delta^n$.)

Base case: $\Lambda^2_1 \longrightarrow \Delta^1$ is $\Xi^2 \longrightarrow \Delta^2$.

Inductive step: We can factorise the spine inclusion as $\Xi^n \hookrightarrow \Lambda^n_k \hookrightarrow \Delta^n$.

We know $\Xi^n \hookrightarrow \Delta^n$ is a trivial cofibration.

$\Xi^n \hookrightarrow \Lambda^n_k$ is also a trivial cofibration by the inductive hypothesis since it can be obtained by filling smaller inner horns.

$\Rightarrow$ $\Lambda^n_k \hookrightarrow \Delta^n$ is a trivial cofibration

2-out-of-3

So, in some sense, $\{\Xi^n \hookrightarrow \Delta^n \mid n \geq 2\} \cup \{(\partial\Delta^n \hookrightarrow \Delta^n) \times (\Delta^0 \hookrightarrow J) \mid n \geq 0\}$ and $\{\Lambda^n_k \hookrightarrow \Delta^n \mid 0 < k < n\}$ are equivalent.

(Indeed, one can use the latter to define the Joyal model structure too.)

Inner horns are definitely more convenient for combinatorics. — the sort of computation where one explicitly deals with simplices

Intuitively, horn-fillers are smaller building blocks than spine-fillers, so the former fits well in more places.

Spines are simpler, so:

- easier to see what they are doing; and
- easier to construct as colimits of simplices

They are preferred especially when using simplicial spaces — complete Segal spaces (rather than simplicial sets) to model ∞-categories; in this case, we don't need $(-) \hat{\times} (\partial J \hookrightarrow J)$ because uniqueness up to coherent homotopy is encoded as:

$$\text{space of maps } \Xi^n \longrightarrow X \quad \simeq \quad \text{space of maps } \Delta^n \longrightarrow X.$$

Similarly, transfer along homotopies is encoded as

$$\text{space of maps } \Delta^0 \to X \quad \simeq \quad \text{space of maps } J \to X.$$