Lecture 5: Quasi-categories via spines (corrected)

Quasi-categories are categories with equalities replaced by coherent homotopies

Recall the nerve functor $N : \underline{Cat} \to \underline{sSet}$ given by

$$(N\mathcal{C})_n = \underline{Cat}([n], \mathcal{C}) = \{\text{commutative } n\text{-simplices in } \mathcal{C}\}.$$

We can characterise the essential image of N using spines.

$\underline{\text{Def}}$: Fix $n \geq 1$. For $1 \leq i \leq n$, write $\eta_i : [1] \longrightarrow [n]$.

$$0 \longmapsto i-1$$
$$1 \longmapsto i$$

(not standard notation)

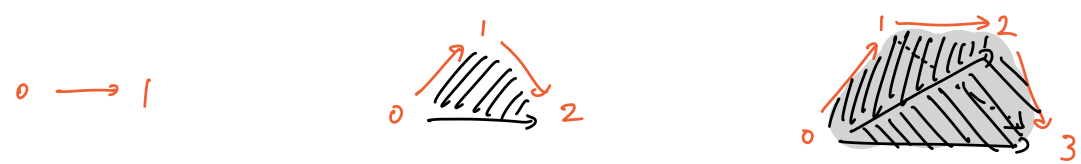
Then the spine of Δ^n is the union of η_i's.

More precisely, it is the simplicial subset Ξ^n of Δ^n given by

$$(\Xi^n)_m = \{\alpha : [m] \to [n] \mid \alpha \text{ factors through } \eta_i : [1] \to [n] \text{ for some } i\}.$$

e.g.

$\Xi^1 = \Delta^1$	$\Xi^2 \subset \Delta^2$	$\Xi^3 \subset \Delta^3$



Fact: $X \in \underline{sSet}$ is of the form $X \cong N\mathcal{C}$ for some $\mathcal{C} \in \underline{Cat}$ iff

$$\forall n \geq 1 \qquad \begin{tikzcd} \Xi^n \arrow[r, "\forall"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

(The "only if" direction is just

$\Xi^n \longrightarrow N\mathcal{C} \rightsquigarrow$ composable sequence of n morphisms in $\mathcal{C}$

$\Updownarrow$

$\Delta^n \rightarrow N\mathcal{C} \leftrightsquigarrow$ commutative n-simplex in $\mathcal{C}$)

Homotopifying this characterisation yields the notion of quasi-category.

replacing uniqueness up to equality
by uniqueness up to coherent homotopy.

What should we mean by a homotopy in this context?

When we were using simplicial sets to model spaces, a homotopy
$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "?"] & Y \end{tikzcd}$$
was a map $H: X \times \Delta^1 \to Y$.

But now we are thinking of simplicial sets as category-like things so $X \times \Delta^1 \to Y$ looks like a natural transformation.

A homotopy $H: f \sim g$ should witness that f & g are similar, so the closest notion in category theory is that of natural isomorphism.

<u>Def</u>: We write J for the nerve of $\{0 \cong 1\} \in \underline{\mathcal{C}at}$, & ∂J for $sk_0(J) \cong \Delta^0 \amalg \Delta^0$.

A homotopy $H: f \sim g$ between $f, g: X \to Y$ in $\underline{sSet}$ is
$$H: X \times J \longrightarrow Y$$
s.t. $H(-, 0) = f$ & $H(-, 1) = g$.

Now let's make precise :
$$\begin{tikzcd} S \arrow[r, "\forall"] \arrow[d, "i"'] & X \\ T \arrow[ur, dashed] & \end{tikzcd}$$
$\exists$ unique up to coherent homotopy

There are three parts : (1) existence $\left(\text{which is just : } \begin{tikzcd} S \arrow[r, "\forall"] \arrow[d, "i"'] & X \\ T \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \right)$

(2) uniqueness up to homotopy

(3) coherence of homotopies

For (2), let's first rephrase the usual uniqueness as follows :

given $f, g : T \to X$ with $fi = gi$, we have $f = g$.

Now replace equalities by homotopies :

given $f, g : T \to X$ with $fi \sim gi$, we can extend this homotopy to $f \sim g$.

(2) Given $f, g : T \to X$ with $f_i \sim g_i$, we can extend this homotopy to $f \sim g$.

$T \times \partial J \to X$ $\qquad$ $S \times J \to X$ $\qquad$ $T \times J \to X$

We can express this statement as:

$$\begin{tikzcd}
(T \times \partial J) \coprod_{S \times \partial J} (S \times J) \arrow[r, "\forall"] \arrow[d] & X \\
T \times J \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

where this map, denoted as $(S \xhookrightarrow{i} T) \hat{\times} (\partial J \hookrightarrow J)$, is induced by:

$$\begin{tikzcd}
S \times \partial J \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & S \times J \arrow[d] \arrow[ddr, bend left] & \\
T \times \partial J \arrow[r] \arrow[drr, bend right] & \bullet \arrow[dr, dashed] & \\
& & T \times J
\end{tikzcd}$$

(It's called the Leibniz product / pushout product of $S \xhookrightarrow{i} T$ & $\partial J \hookrightarrow J$.)

(3) "a coherence homotopy for each equality that holds in the strict case" translates to:

the homotopy $f \sim g$ is unique up to homotopy

this is unique up to homotopy

this is ...

This amounts to asking for lifts:
$$\begin{tikzcd} \cdot \arrow[r, "\forall"] \arrow[d] & X \\ \cdot \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$
for repeated Leibniz products

$$(S \xrightarrow{i} T) \hat{\times} (\partial J \hookrightarrow J) \hat{\times} \cdots \hat{\times} (\partial J \hookrightarrow J)$$

<u>Def</u>: For a set $\mathcal{U}$ of morphisms in <u>sSet</u>, we write:

$$\overline{\mathcal{U}} = \left\{ (S \xrightarrow{i} T) \hat{\times} (\partial J \hookrightarrow J)^{\hat{\times} n} \;\middle|\; i \in \mathcal{U},\ n \geq 0 \right\}$$

often denoted $\Lambda^{\infty}(\mathcal{U})$

<u>Def</u>: A quasi-category is a simplicial set X s.t.
$$\begin{tikzcd} \cdot \arrow[r, "\forall"] \arrow[d, "f"'] & X \\ \cdot \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$
for all $f \in \overline{\{ \Xi^n \hookrightarrow \Delta^n \mid n \geq 2 \}} \cup \{ (\partial\Delta^n \hookrightarrow \Delta^n) \hat{\times} (\Delta^0 \hookrightarrow J) \mid n \geq 0 \}$.

any spine in X can be completed to a unique simplex up to coherent homotopy.

can transfer things along homotopies.

<u>Fact</u>: $N(\mathcal{C})$ is a quasi-category for any $\mathcal{C} \in$ <u>Cat</u>.

The RLP w.r.t. $(\partial\Delta^n \hookrightarrow \Delta^n) \hat{\times} (\Delta^0 \hookrightarrow J)$ says :

given n-simplex + homotopy on bdry, can extend it to homotopy on n-simplex.

$$
\begin{tikzcd}
\partial\Delta^n \times \Delta^0 \arrow[r] \arrow[d] & \partial\Delta^n \times J \arrow[d] \arrow[ddr, bend left] & \\
\Delta^n \times \Delta^0 \arrow[r] \arrow[drr, bend right] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
 & & \Delta^n \times J
\end{tikzcd}
$$

i.e. We can transfer n-simplices along homotopies.

The following Joyal model structure on $\underline{sSet}$ captures the homotopy theory of quasi-categories:

- cofibrations = monomorphisms ⇝ every object is cofibrant
- $\{ \Xi^n \hookrightarrow \Delta^n \mid n \geq 2 \} \cup \{ (\partial\Delta^n \hookrightarrow \Delta^n) \hat{\times} (\Delta^0 \hookrightarrow J) \mid n \geq 0 \}$ generates trivial cofibrations $\mathcal{C} \cap \mathcal{W}$ in a suitable sense; in particular, fibrant objects = quasi-categories.

(Both quasi-categories & the Joyal model structures are usually defined using inner horns rather than spines. Come to Lecture 6 for more.)

<u>Fact</u>: $X \times J$ is a cylinder object for any $X \in \underline{sSet}$ in this model str.

So, given a map f between quasi-categories,

$$f \in \mathcal{W} \iff f \text{ is a homotopy equivalence w.r.t. } J.$$

Since homotopies w.r.t. J are like natural isomorphisms, homotopy equivalences w.r.t. J are like equivalences of categories.

Recall the homotopy coherent nerve functor

$$N_{hc} : \underline{sSet}\text{-}\underline{Cat} \longrightarrow \underline{sSet}$$

$$\mathcal{K} \longmapsto \left([n] \mapsto \underline{sSet}\text{-}\underline{Cat}\left(\mathfrak{C}[n], \mathcal{K}\right)\right)$$

homotopy coherent simplex given by

$$\hom_{\mathfrak{C}[n]}(i,j) = \begin{cases} \underbrace{\Delta^1 \times \dots \times \Delta^1}_{j-i-1} & i \leq j \\ \emptyset & i > j \end{cases}$$

Fact: If $\mathcal{K}(X,Y)$ is a Kan complex for all $X, Y \in \mathcal{K}$, then $N_{hc}(\mathcal{K})$ is a quasi-category.

full simplicial subcat. of $\underline{sSet}$

e.g. $N_{hc}(\underline{Kan})$ is a quasi-category. This is the "∞-category of spaces", and it's the homotopical counterpart of the ordinary category $\underline{Set}$.

In fact, $\underline{sSet} \underset{N_{hc}}{\overset{\mathfrak{C}}{\rightleftarrows}} \underline{sSet}\text{-}\underline{Cat}$ ($\perp$) becomes a Quillen equivalence if we equip:

- $\underline{sSet}$ with Joyal model str.
- $\underline{sSet}\text{-}\underline{Cat}$ with Bergner model str.

In the Bergner model structure on $\underline{sSet}$-$\underline{Cat}$,

- $F : \mathcal{K} \to \mathcal{L}$ is in $\mathcal{W}$ iff
  - it's surjective on objects up to homotopy equivalence; and
  - $F_{X,Y} : \mathcal{K}(X,Y) \to \mathcal{L}(FX,FY)$ is a weak homotopy equivalence
- $\mathcal{K}$ is fibrant iff $\mathcal{K}(X,Y)$ is a Kan complex for all $X, Y \in \mathcal{K}$.
- $\mathcal{K}$ is cofibrant iff it's free in a suitable sense.

So (the Bergner model structure & hence) the Joyal model structure captures the homotopy theory of space-enriched categories.

In this lecture, we described quasi-categories & the Joyal model structure as a particular instance of general theory due to Cisinski. This theory takes: $\boxed{\text{Extra}}$

- category $\mathcal{C}$ (Δ in our case)
- object $J \in [\mathcal{C}^{op}, \underline{Set}]$ representing homotopies ($J = N(0 \cong 1)$)
- algebraic operations + conditions encoded as maps in $[\mathcal{C}^{op}, \underline{Set}]$

$$\left(\begin{array}{ll} \Xi^2 \hookrightarrow \Delta^2 & \text{binary composition} \\ \Xi^3 \hookrightarrow \Delta^3 & \text{ternary composition / associativity} \\ \vdots & \end{array}\right)$$

& gives back a model structure on $[\mathcal{C}^{op}, \underline{Set}]$ in which:

- cofibrations = monomorphisms
- homotopies are defined w.r.t. J.
- fibrant objects = objects in which:
  - we have operations subject to conditions as above defined / satisfied up to coherent homotopy
  - we can transfer things along homotopies.

(It's actually more general.)