Lecture 4: Homotopy coherent nerve

∞-functors should preserve homotopies rather than equalities

We want ∞-categories to be category-like structures that can deal with homotopies.

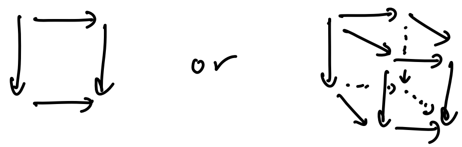
So maybe we can take : ∞-categories = model categories ?

This works for some purposes, but obviously ∞-functors $\neq$ Quillen adjunctions.

We want to define ∞-categories in such a way that the natural notion of morphism we (automatically) obtain is that of ∞-functor.

So let's think about what an ∞-functor $F : \mathcal{C} \to \mathcal{D}$ should do.

In particular, when $\mathcal{C}$ is something really simple like



,

what should commutative squares/cubes in $\mathcal{D}$ look like?

Recall the fundamental group:

$$\pi_1(X, x) = \{ f : I \longrightarrow X \text{ in } \underline{\mathrm{Top}} \mid f(0) = f(1) = x \} / \text{endpoint-preserving homotopy}$$

Here we're quotienting by homotopy to make $\pi_1(X, x)$ into a group. set-based str.

But really the natural thing to consider in the space-based context is ...

<u>Def</u>: Let $X \in \underline{\mathrm{Top}}$ and $x \in X$. The loop space on the pair (X, x) is the set

we'll write ΩX $\quad \Omega(X, x) = \{ f : I \longrightarrow X \text{ in } \underline{\mathrm{Top}} \mid f(0) = f(1) = x \}$

equipped with suitable topology.

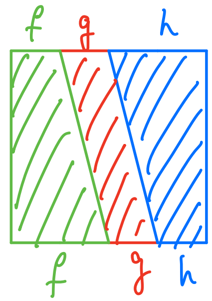
We still have the multiplication $(g * f)(s) = \begin{cases} f(2s) & \text{if } 0 \leq s \leq 1/2, \\ g(2s-1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$ but it's NOT

associative in the usual sense; $(h * g) * f$ & $h * (g * f)$ are homotopic as maps $I \to X$,

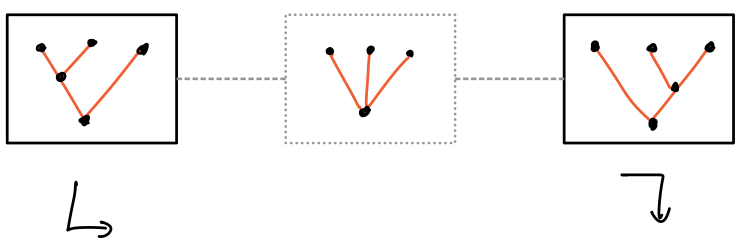
which translates to a path in ΩX.

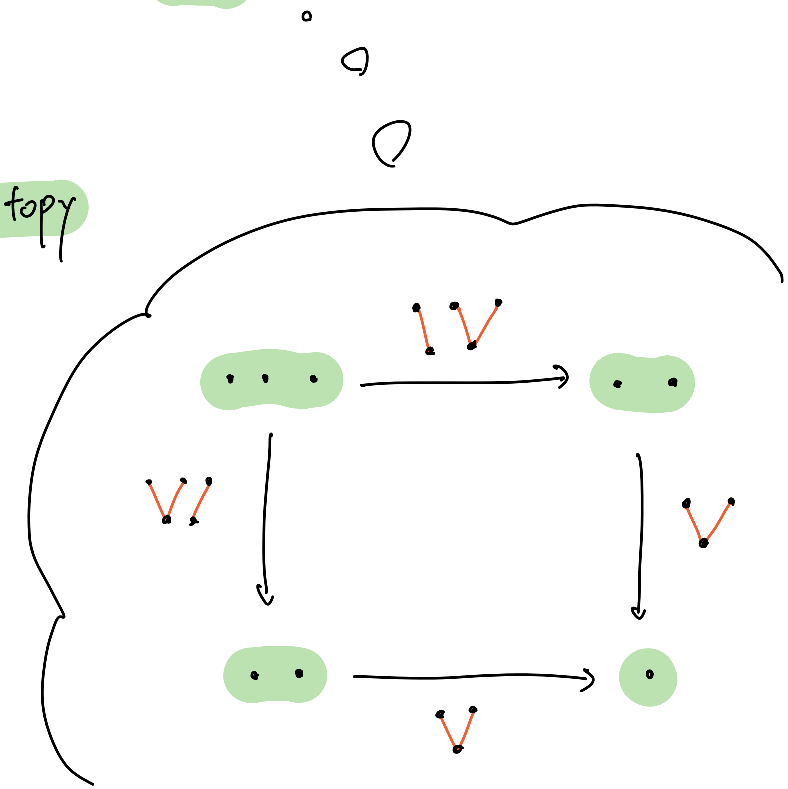
These paths assemble into a homotopy which witnesses associativity.

$$\begin{tikzcd}
\Omega X \times \Omega X \times \Omega X \arrow[rr, "\mathrm{id} \times *"] \arrow[dd, "* \times \mathrm{id}"'] & & \Omega X \times \Omega X \arrow[dd, "*"] \\
& h, g, f \arrow[r, mapsto] \arrow[d, mapsto] & h, g * f \arrow[d, mapsto] \\
& & h * (g * f) \\
& & \wr \\
& h * g, f \arrow[r, mapsto] & (h * g) * f \\
\Omega X \times \Omega X \arrow[rr, "*"'] & & \Omega X
\end{tikzcd}$$

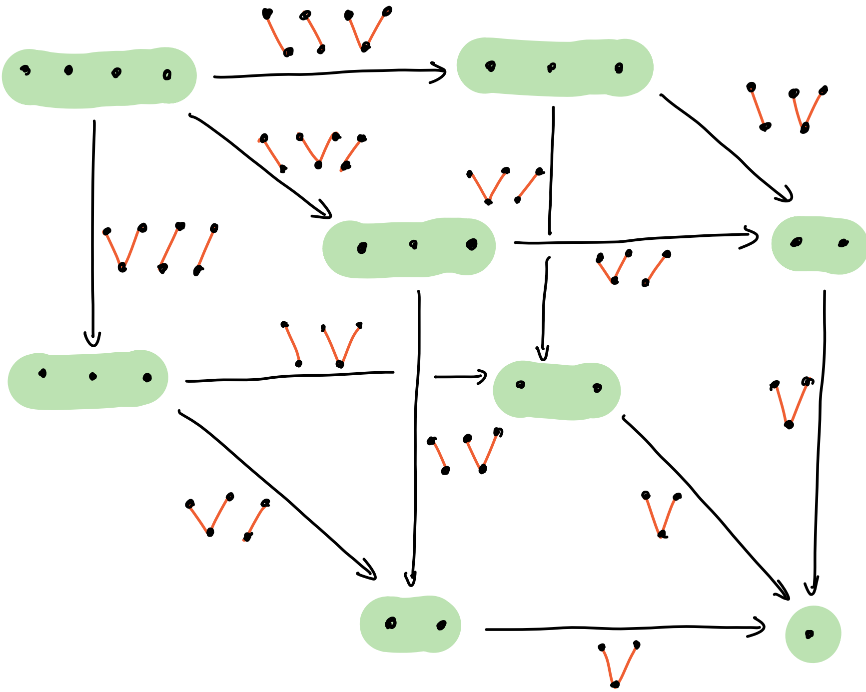


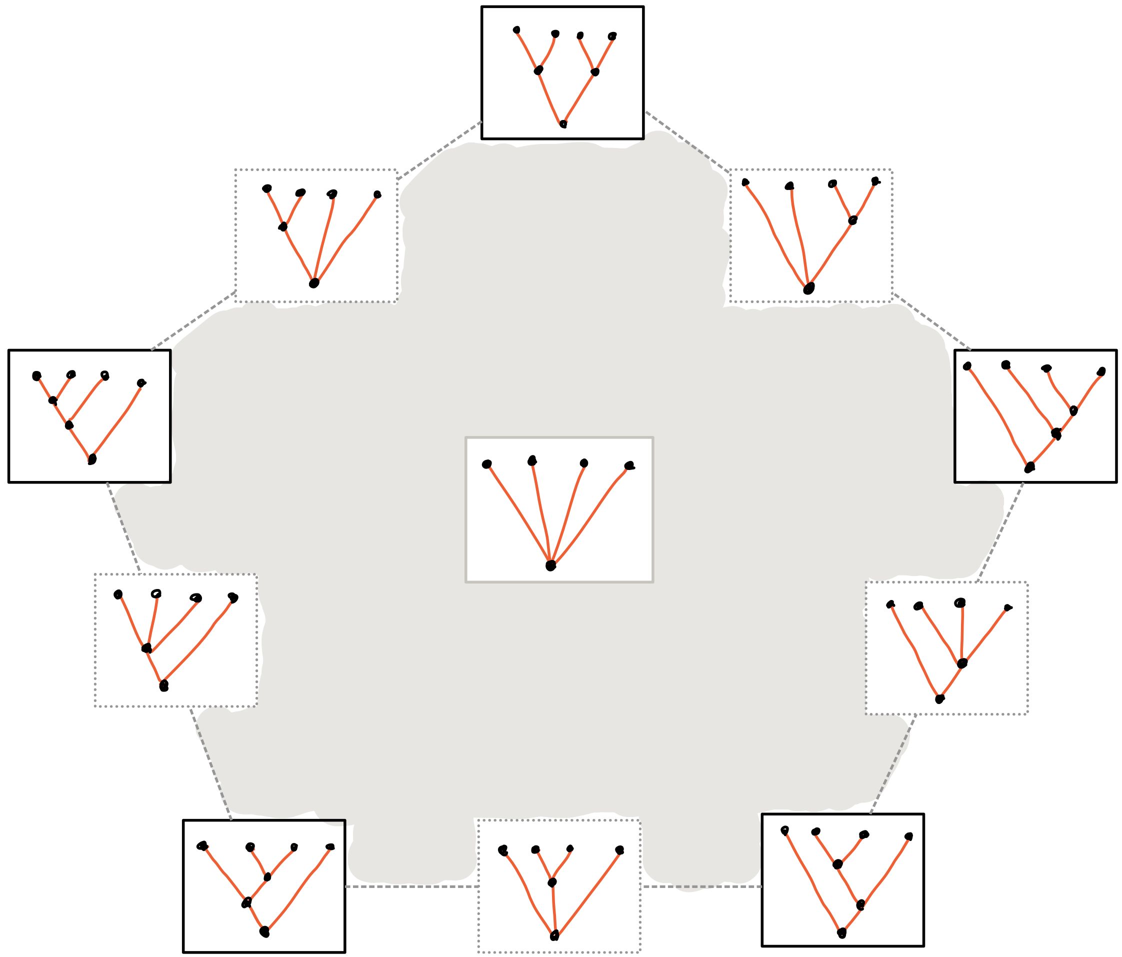
We can visualise the composites as trees and the homotopy as a deformation of one tree into the other:





We also have a homotopy between homotopies witnessing coherence of associativity.





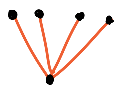
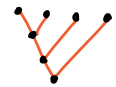
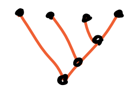
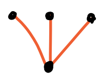
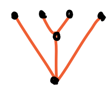
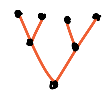
and homotopies between homotopies between homotopies etc.

Why do we want coherence?

An equality "$A = B$" is useful because we can transfer our understanding of A to B along it & vice versa.

When dealing with spaces, we are using homotopies instead of equalities, so we want the homotopies to be useful in the same way.

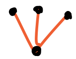
we want to transfer things along homotopies without worrying about anything.

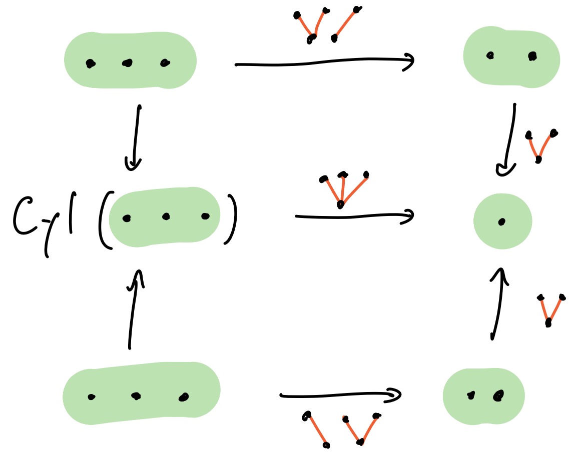
This particular coherence  witnesses that, the two paths from  to  obtained using instances of  (via  & via ) are equivalent so that we can transfer things without worrying about which path we took.

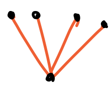
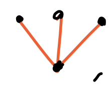
In general, we want a coherence homotopy for each equality that is (trivially) satisfied in the strict case. This ensures that the homotopies behave as much like equalities as possible without actually being equalities.

So we want ∞-functors to preserve (coherent) homotopies.

Model categories are not well suited for supporting such maps:

e.g. The associativity  looks like:



The coherence  involves taking five instances of , composing them using more cylinder objects, which are then connected via $\mathrm{Cyl}(\mathrm{Cyl}(\bullet\ \bullet\ \bullet\ \bullet))$.

(If we are only interested in mapping small categories $\mathcal{C}$ into $\mathcal{M}$, we can talk about e.g. homotopy co/limits by considering suitable model structures on $[\mathcal{C}, \mathcal{M}]$.)

Working in $\mathrm{Ho}(\mathcal{M})$ is no good either because we can't talk about coherence.

Maybe the problem was the lack of direct access to homotopies.

We can solve this using enrichment.

<u>Def</u>: Let $\mathcal{V}$ be a category with finite products.

A $\mathcal{V}$-enriched category $\mathcal{A}$ is comprised of:

- a collection $ob(\mathcal{A})$ of objects
- hom-object $\hom_{\mathcal{A}}(A,B) \in \mathcal{V}$ for $A, B \in ob(\mathcal{A})$
- composition $\hom_{\mathcal{A}}(B,C) \times \hom_{\mathcal{A}}(A,B) \to \hom_{\mathcal{A}}(A,C)$ for $A, B, C \in ob(\mathcal{A})$
- unit $1 \to \hom_{\mathcal{A}}(A,A)$ for $A \in ob(\mathcal{A})$

satisfying unit & associative laws.

More generally, we can enrich over any monoidal category $\mathcal{V}$.

e.g. $\underline{sSet}$ is enriched over itself; the internal hom $[X,Y] \in \underline{sSet}$ is given by

$$[X,Y]_n = \underline{sSet}(X \times \Delta^n, Y).$$

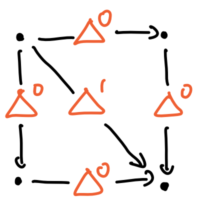
In particular, $[X,Y]_0 \cong \underline{sSet}(X,Y)$ & $[X,Y]_1$ is the set of homotopies.

$\underline{Top}$ can also be enriched over $\underline{sSet}$ by $\hom_{\underline{Top}}(X,Y) = \mathrm{Sing}([X,Y])$ internal hom in $\underline{Top}$.

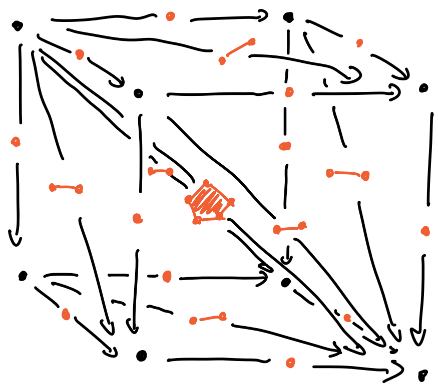
Both sSet & Top are simplicial model categories, which essentially means that the simplicial enrichment & the model str. capture the same homotopy theory.

So if we set ∞-categories = sSet-enriched categories, at least we can directly talk about homotopies.

However, to get homotopy coherent squares/cubes, we have to map out of:



or



("$A \xrightarrow{X} B$" means $\mathrm{hom}(A,B) = X \in$ sSet.)

Does specifying the shapes of naturally occurring diagrams have to be this complicated?

Do we still not have enough structure?

Actually, the problem is that we've got too much structure.

Simplicial categories remember equalities between morphisms, so the natural morphisms between them (i.e. simplicial functors) also preserve those equalities.

But if ∞-categories only remembered homotopies (so that equalities are treated just like any other homotopy), then ∞-functors from the ordinary commutative square / cube should correspond precisely to homotopy coherent squares/cubes.

So now our goal is to come up with a way to turn simplicial categories into something that remembers objects, morphisms & (higher) homotopies but not equalities.

The nerve functor $N\colon \underline{Cat} \to \underline{sSet}$ is given by

$$(N\mathcal{C})_n = \underline{Cat}([n], \mathcal{C}).$$

Here the commutative n-simplex $[n]$ has objects $0, 1, \dots, n$ &

$$\hom_{[n]}(i,j) = \begin{cases} * & \text{if } i \le j \\ \emptyset & \text{if } i > j \end{cases}$$

e.g. 2-simplex

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] & & \bullet \end{tikzcd}$$

witnesses equality $gf = h$.

The homotopy coherent nerve functor $N_{hc}\colon \underline{sSet\text{-}Cat} \longrightarrow \underline{sSet}$ is given by

$$(N_{hc}\mathcal{K})_n = \underline{sSet\text{-}Cat}(\mathfrak{C}[n], \mathcal{K}).$$

Here the homotopy coherent n-simplex $\mathfrak{C}[n]$ has objects $0, 1, \dots, n$ &

$$\hom_{\mathfrak{C}[n]}(i,j) = \begin{cases} ? & \text{if } i \le j \\ \emptyset & \text{if } i > j \end{cases}$$

e.g. 2-simplex

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] & & \bullet \end{tikzcd}$$

witnesses homotopy $gf \sim h$.

What should $\hom_{\mathfrak{C}[n]}(i,j)$ be for $i \le j$?

e.g. $i \rightsquigarrow j$,
$i \rightsquigarrow i+1 \rightsquigarrow i+3 \rightsquigarrow j$, etc.

We want nothing to commute strictly, so different paths from i to j should give rise to different morphisms from i to j.

$k \in S \iff$ the path visits k.

0-simplices $\leftsquigarrow\!\rightsquigarrow$ sets S s.t. $\{i,j\} \subset S \subset \{i, i+1, \dots, j\}$

(Composition is given by taking the union.)

These 0-simplices can be arranged into a $(j-i-1)$-dimensional cube.

axes corresponding to $i<k<j$

e.g.

$$\begin{tikzcd}
\{0,3\} \arrow[r, dash, dashed] \arrow[d, dash, dashed] & \{0,1,3\} \arrow[d, dash, dashed] \\
\{0,2,3\} \arrow[r, dash, dashed] & \{0,1,2,3\}
\end{tikzcd}$$

homotopies behave as much like equalities as possible

Now we also want everything to commute up to coherent homotopy, which in this case translates to $\hom_{\mathfrak{C}[n]}(i,j) \simeq \hom_{[n]}(i,j) = *$.

So we set $\hom_{\mathfrak{C}[n]}(i,j) = \underbrace{\Delta^1 \times \dots \times \Delta^1}_{(j-i-1) \text{ times}}$ for $i \le j$.

nerve of
$\{S \mid \{i,j\} \subset S \subset \{i, i+1, \dots, j\}\}$
ordered by inclusion

So, do we really get homotopy coherent squares/cubes easily?

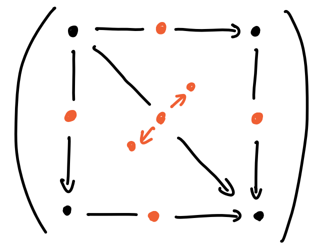
Let's consider $\Delta^1 \times \Delta^1 \longrightarrow N_{hc}(\mathcal{K})$. By adjunction, this transposes to $\mathfrak{C}(\Delta^1 \times \Delta^1) \longrightarrow \mathcal{K}$.

Since $\mathfrak{C}$ sends the pushout

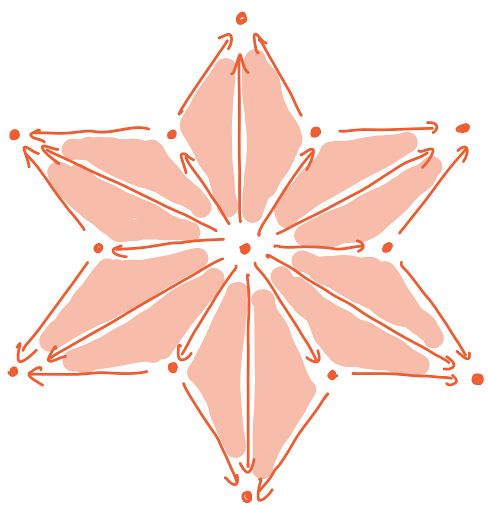
$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[d] & \Delta^2 \arrow[d] \\ \Delta^2 \arrow[r] & \Delta^1 \times \Delta^1 \end{tikzcd}$$

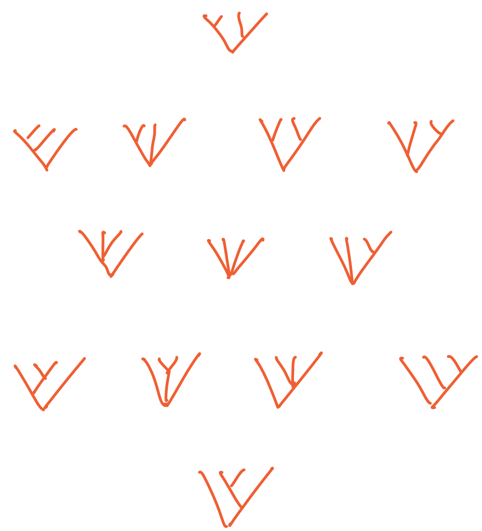
to a pushout

$$\begin{tikzcd} \mathfrak{C}[1] \arrow[r] \arrow[d] & \mathfrak{C}[2] \arrow[d] \\ \mathfrak{C}[2] \arrow[r] & \mathfrak{C}(\Delta^1 \times \Delta^1) \end{tikzcd}$$

where $\mathfrak{C}[1] = (0 \to 1)$ and $\mathfrak{C}[2] = \left(\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}\right)$, we have $\mathfrak{C}(\Delta^1 \times \Delta^1) =$  .

The diagonal hom of $\mathfrak{C}(\Delta^1 \times \Delta^1 \times \Delta^1)$ looks like:



corresponding to  in the case of ΩX