Lecture 3: Model categories

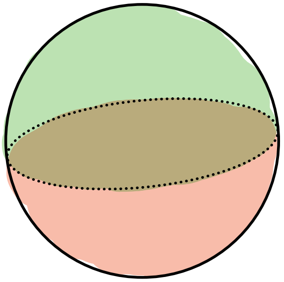
CW-complexes and Kan complexes support equivalent homotopy theories.

Consider :

$$\begin{tikzcd} D^2 \arrow[d, "\wr"'] & S^1 \arrow[l] \arrow[r] \arrow[d, "\wr"] & D^2 \arrow[d, "\wr"] \\ * & S^1 \arrow[l] \arrow[r] & * \end{tikzcd}$$

in $\underline{Top}$.

The vertical maps are all (weak) htpy eq., but if we take the pushout of each row, we get non-equivalent spaces :



$\not\simeq \quad * .$

Using the language of model categories, we can deal with these (seemingly) ill-behaved equivalences and make precise e.g. S^2 is the "right" pushout.

Def: Let $\mathcal{M}$ be a category with small limits & colimits.

A model structure on $\mathcal{M}$ consists of three classes of morphisms called:

- weak equivalences $\mathcal{W}$ $(\xrightarrow{\sim})$ } determine the homotopy theory.
- cofibrations $\mathcal{C}$ $(\rightarrowtail)$
- fibrations $\mathcal{F}$ $(\twoheadrightarrow)$

} facilitate computation.

satisfying (1 - 4).

Weak equivalences are the equivalences we care about (e.g. weak htpy eq. in Top/sSet) and the axioms ensure that they indeed behave like equivalences.

(1) If

$$\begin{tikzcd}
{} \arrow[r, "u"] \arrow[rr, bend left, "id"] \arrow[d, "g"'] & {} \arrow[r, "r"] \arrow[d, "f"] & {} \arrow[d, "g"] \\
{} \arrow[r, "v"] \arrow[rr, bend right, "id"'] & {} \arrow[r, "s"] & {}
\end{tikzcd}$$

and $f \in \mathcal{W}$ then $g \in \mathcal{W}$.

if $\mathcal{W} = \{iso\}$ then $g^{-1} =$

$$\begin{tikzcd}
& {} \arrow[r, "r"] & {} \\
{} \arrow[r, "v"'] & {} \arrow[u, "f^{-1}"'] &
\end{tikzcd}$$

(2) Given $\xrightarrow{f}\xrightarrow{g}$, if two of f, g, gf are in $\mathcal{W}$ then so is the last.

2-out-of-3 property

Although we are only interested in CW-complexes / Kan complexes, these objects are NOT closed under the sort of things we want to do (e.g. taking (co)limits in Top / sSet).

So we'll work in Top / sSet & use the model structure to capture the "niceness" of CW-cx / Kan cx.

- CW-complexes are built as colimits (of disks), so it's easy to construct maps out of them.
- Kan complexes have the dual property; it's easy to construct maps into them (by definition).

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\forall"] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

In the language of model categories,

- CW-complexes are cofibrant, and
- Kan complexes are fibrant.

precise definitions in a few slides

Cofibrations are maps with a relative version of easy-to-map-out-of property.

i.e. it's easy to extend maps along them.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, tail] & \bullet \\
\bullet \arrow[ur, dashed] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \bullet \arrow[d, two heads] \\
\bullet \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}$$

Dually, fibrations have a relative version of easy-to-map-into property.

i.e. it's easy to lift maps through them.

can extend a map defined on a subset to the whole set

can always pick a point in the preimage

Cofibrations are "like injections" and fibrations are "like surjections".

Def: Let f, g be morphisms in $\mathcal{M}$. If every commutative square of the form

$$\begin{tikzcd} {} \arrow[r] \arrow[d, "f"'] & {} \arrow[d, "g"] \\ {} \arrow[r] \arrow[ur, dashed] & {} \end{tikzcd}$$

simultaneously extending the unique composite ↘ along f and lifting it through g.

admits a diagonal lift, we say:

- f has the left lifting property (LLP) w.r.t. g, and
- g has the right lifting property (RLP) w.r.t. f.

trivial / acyclic fibrations

(3) $\mathcal{C} = \{$maps with LLP w.r.t. all maps in $\mathcal{F} \cap \mathcal{W}\}$.

$$\begin{tikzcd} {} \arrow[r] \arrow[d, tail] & {} \arrow[d, two heads, "\sim"'] \\ {} \arrow[r] \arrow[ur, dashed] & {} \end{tikzcd}$$

these lifts "trivially" exist up to htpy.

trivial / acyclic cofibrations

$\mathcal{F} = \{$maps with RLP w.r.t. all maps in $\mathcal{C} \cap \mathcal{W}\}$.

$$\begin{tikzcd} {} \arrow[r] \arrow[d, tail, "\sim"'] & {} \arrow[d, two heads] \\ {} \arrow[r] \arrow[ur, dashed] & {} \end{tikzcd}$$

(It follows from (1) & (3) that $\mathcal{C} \cap \mathcal{W}$ & $\mathcal{F} \cap \mathcal{W}$ can be similarly characterised by lifting properties.)

(4) Any map f in $\mathcal{M}$ admits factorisations of the form :

$$\begin{tikzcd} & {} \arrow[dr, two heads, "\in \mathcal{F}"] & \\ {} \arrow[ur, hook, "\sim"', "\in \mathcal{C}\cap\mathcal{W}"] \arrow[rr, "f"'] & & {} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & {} \arrow[dr, two heads, "\sim"', "\in \mathcal{F}\cap\mathcal{W}"] & \\ {} \arrow[ur, hook, "\in \mathcal{C}"] \arrow[rr, "f"'] & & {} \end{tikzcd}$$

approximation of f by co/fibration.

In most cases, these factorisations can be chosen functorially : $\mathcal{M}^{\{\cdot \to \cdot\}} \longrightarrow \mathcal{M}^{\{\cdot \to \cdot \to \cdot\}}$

Examples :

$\mathcal{M}$	Top classical / Quillen	sSet classical / Kan / Quillen	Cat canonical / folk
$\mathcal{W}$	weak htpy eq.	weak htpy eq.	equivalences
$\mathcal{F}$	RLP wrt $D^n \times \{0\} \to D^n \times I$ Serre fibrations	RLP wrt $\Lambda^n_k \to \Delta^n$ Kan fibrations	RLP wrt $\{\bullet\} \to \{\bullet \cong \bullet\}$ isofibrations
$\mathcal{C}$	LLP wrt $\mathcal{F} \cap \mathcal{W}$	monomorphisms	injective on objects

Def: $X \in \mathcal{M}$ is

- cofibrant if $0 \xrightarrow{!} X$ is a cofibration — easy to map out of
  e.g. CW-cx in Top
- fibrant if $X \xrightarrow{!} 1$ is a fibration — easy to map into
  e.g. Kan cx in sSet

A cofibrant replacement of $X \in \mathcal{M}$ is cofibrant $QX \in \mathcal{M}$ t/w $QX \xrightarrow{\sim} X$.

A fibrant replacement ______ fibrant $RX \in \mathcal{M}$ t/w $X \xrightarrow{\sim} RX$.

Observation: For any $X \in \mathcal{M}$, by (4) we can always construct

$$\begin{tikzcd} & QX \arrow[dr, two heads, "\wr"] & \\ 0 \arrow[ur, tail] \arrow[rr, "!"'] & & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & RX \arrow[dr, two heads] & \\ X \arrow[ur, tail, "\sim"] \arrow[rr, "!"'] & & 1 \end{tikzcd}$$

⇝ • Can extend Q & R to morphisms, e.g.

$$\begin{tikzcd} 0 \arrow[rr, "!"] \arrow[d, tail] & & QY \arrow[d, two heads, "\wr"] \\ QX \arrow[r, two heads, "\sim"] \arrow[urr, dashed, "Qf"] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

In general, Q & R are functorial only up to htpy.

• Any $X \in \mathcal{M}$ is connected by a zigzag of weak eq. to an object that is both fibrant & cofibrant:

$$\begin{tikzcd} 0 \arrow[r, tail] & QX \arrow[r, two heads, "\sim"] \arrow[d, tail, "\wr"] & X \\ & RQX \arrow[d, two heads] & \\ & 1 & \end{tikzcd}$$

In a model category, the notion of homotopy makes sense.

Def: Let $X, Y \in \mathcal{M}$.

- A cylinder object for X is a factorisation

$$\begin{tikzcd} & Cyl(X) \arrow[dr, "\sim"] & \\ X \amalg X \arrow[ur, tail, "\langle i_0, i_1 \rangle"] \arrow[rr, "\langle id, id \rangle"'] & & X \end{tikzcd}$$

e.g. $X \times I$ in Top, $X \times \Delta^1$ in sSet.

- A path object for Y is a factorisation

$$\begin{tikzcd} & Path(Y) \arrow[dr, two heads, "(p_0, p_1)"] & \\ Y \arrow[ur, "\sim"] \arrow[rr, "(id, id)"'] & & Y \times Y \end{tikzcd}$$

.

Def: Let $f, g : X \to Y$ in $\mathcal{M}$.

- A left homotopy from f to g is (a choice of $Cyl(X)$ t/w) a map $Cyl(X) \xrightarrow{H} Y$ s.t. $Hi_0 = f$ & $Hi_1 = g$. $f \overset{l}{\simeq} g$

- A right homotopy from f to g is (a choice of $Path(Y)$ t/w) a map $X \xrightarrow{K} Path(Y)$ s.t. $p_0 K = f$ & $p_1 K = g$. $f \overset{r}{\simeq} g$

Observation:

$f \overset{l}{\simeq} g \Rightarrow hf \overset{l}{\simeq} hg$

$f \overset{r}{\simeq} g \Rightarrow fk \overset{r}{\simeq} gk$

for

$$\begin{tikzcd} \cdot \arrow[r, "k"] & \cdot \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \cdot \arrow[r, "h"] & \cdot \end{tikzcd}$$

Fact: If $X \in \mathcal{M}$ is cofibrant and $Y \in \mathcal{M}$ is fibrant, then $\overset{l}{\sim}$ & $\overset{r}{\sim}$ agree on $\mathcal{M}(X,Y)$, and moreover it's an equivalence relation. We'll just write $\sim$ in this case.

Whitehead's theorem:
Let $f: X \to Y$ with X & Y both fibrant & cofibrant.
Then $f \in \mathcal{W}$ iff f is a homotopy eq. $\exists g: Y \to X$ s.t. $gf \sim id_X$ & $fg \sim id_Y$.

Def: The homotopy category $Ho(\mathcal{M})$ is obtained from $\mathcal{M}$ by

- restricting to the fibrant & cofibrant objects, and
- quotienting the hom-sets by $\sim$.

Fact:

- There is a functor $\gamma : \mathcal{M} \longrightarrow Ho(\mathcal{M})$ given by
$$\gamma(X \xrightarrow{f} Y) = (RQX \xrightarrow{[RQf]} RQY)$$
htpy class containing RQf.
- $f \in \mathcal{W} \Leftrightarrow \gamma(f)$ is an isomorphism.
- γ is universal among those functors $\mathcal{M} \to \mathcal{C}$ that send weak equivalences to isomorphisms:
$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\gamma"] \arrow[dr] & Ho(\mathcal{M}) \arrow[d, dashed, "\cong"', "\text{essentially unique}"] \\
& \mathcal{C}
\end{tikzcd}$$

$Ho(\mathcal{M})$ is the localisation of $\mathcal{M}$ at $\mathcal{W}$.

What's the right notion of morphism between model categories?

If we just want $F : \mathcal{M} \to \mathcal{N}$ to induce $Ho(\mathcal{M}) \to Ho(\mathcal{N})$ then F simply needs to preserve weak eq., but such F can mess up computations.

Recall: co/fibrations facilitate computations

Here's a hint.

Exercise: Consider an adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ with $\perp$ between the arrows, and morphisms $f : A \to B$ in $\mathcal{C}$ & $g : X \to Y$ in $\mathcal{D}$. Then there is a bijection

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & GX \arrow[d, "Gg"] \\ B \arrow[r] & GY \end{tikzcd} \text{ in } \mathcal{C} \quad \leftrightsquigarrow \quad \begin{tikzcd} FA \arrow[r] \arrow[d, "Ff"'] & X \arrow[d, "g"] \\ FB \arrow[r] & Y \end{tikzcd} \text{ in } \mathcal{D}$$

and either admits a diagonal lift iff the other does. Consequently,

f has LLP w.r.t. Gg $\Leftrightarrow$ Ff has LLP w.r.t. g.

Def: An adjunction between model categories $$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "G"] \end{tikzcd}$$ is a Quillen adjunction if

- F preserves cofibrations ; and
- G preserves fibrations.

G preserves triv. fib. ⇔ (F preserves cofibrations)

F preserves triv. cof. ⇔ (G preserves fibrations)

F is left Quillen / G is right Quillen

Quillen adjunctions present homotopically meaningful universal properties.

Fact: A Quillen adjunction $$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "G"] \end{tikzcd}$$ induces an adjunction $$\begin{tikzcd} Ho(\mathcal{M}) \arrow[r, bend left, "LF"] \arrow[r, phantom, "\perp"] & Ho(\mathcal{N}) \arrow[l, bend left, "RG"] \end{tikzcd}$$

derived functors

e.g. $$\begin{tikzcd} \underline{sSet} \arrow[r, bend left, "|\cdot|"] \arrow[r, phantom, "\perp"] & \underline{Top} \arrow[l, bend left, "Sing"] \end{tikzcd}$$. In fact, it's an example of a Quillen equivalence.

Def: A Quillen equivalence is a Quillen adjunction $F \dashv G$ s.t.

$$f : X \to GY \in \mathcal{W}_{\mathcal{M}} \iff \bar{f} : FX \to Y \in \mathcal{W}_{\mathcal{N}}$$

(transpose of f

Fact: $LF \dashv RG$ is an adjoint equivalence iff $F \dashv G$ is a Quillen equivalence.