Lecture 2: Spaces as Kan complexes

By "spaces", we mean Kan complexes up to (weak) homotopy equivalence.

Last lecture: By spaces, we mean CW-complexes up to (weak) homotopy equivalence.

detected by htpy groups

simplicial set (today)

$$\begin{tikzcd} & \mathrm{Sing}(X) \arrow[dr, squiggly] & \\ X \arrow[ur, squiggly] \arrow[rr, squiggly] & & \pi_n(X)\text{'s} \end{tikzcd}$$

CW-cx   htpy groups

$\pi_0(X) = \{\text{path components of } X\}$

Fact: If we are given $f: X \to Y$ between CW-cx s.t. $\pi_n(f)$ invertible for all n then we know that f is a homotopy equivalence, but we CANNOT deduce the existence of such f from isomorphisms $\pi_n(X) \cong \pi_n(Y)$.

To specify a space, we have to know how holes of different dimensions fit together.

Recall: Δ is the category of totally ordered sets $[n] = \{0 < 1 < \ldots < n\}$ and order-preserving maps.

The category of simplicial sets is $\underline{sSet} = [\Delta^{op}, \underline{Set}]$.

The representable $\Delta(-, [n])$ is denoted by Δ^n.

Notation / terminology:

For $X \in \underline{sSet}$ and $n \geq 0$, we write $X_n = X([n])$.

An element $x \in X_n$ is called an n-simplex in X.

By Yoneda, $x \in X_n \leftrightsquigarrow \Delta^n \to X$.

$X(\alpha) : X_n \to X_m$

For $\alpha : [m] \to [n]$, we denote the image of $x \in X_n$ under the action of α by $x \cdot \alpha = (X(\alpha))(x)$.

consistent with: $\Delta^m \xrightarrow{\alpha} \Delta^n \xrightarrow{x} X$

we also have $(x \cdot \alpha) \cdot \beta = x \cdot (\alpha\beta)$

We'll think of $[n] = \{0 < 1 < \dots < n\}$ as a combinatorial representation of the topological n-simplex $|\Delta^n|$, but "directed" i.e. equipped with ordering on vertices

e.g.	$[0]$	$[1]$	$[2]$	$[3]$

Let's compare these pictures to the simplices in $\Delta^n = \Delta(-, [n])$.

Δ^1 has 0-simplices $[0] \rightrightarrows [1]$ corresponding to the endpoints

$0 \mapsto 0$

$0 \mapsto 1$

1-simplex $[1] \xrightarrow{id} [1]$ ——— the interior

appear in the picture.

but it also has 1-simplex $[1] \longrightarrow [1]$

$0 \mapsto 0$

$1 \mapsto 0$

2-simplex $[2] \longmapsto [1]$ etc.

$0 \mapsto 0$

$1 \mapsto 1$

$2 \mapsto 1$

???

Def: A morphism in Δ is called degenerate if it is NOT injective.

The simplices that appear in the pictures are precisely the non-degenerate ones:

= injective

e.g. $[1] \longrightarrow [2]$

$0 \mapsto 0$

$1 \mapsto 1$

2

corresponds to the edge

$[2] \longrightarrow [3]$

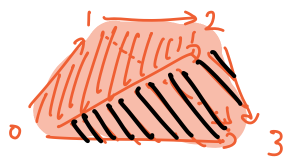
$0 \mapsto 0$

1

$1 \mapsto 2$

$2 \mapsto 3$

corresponds to the face



Def: Let $X \in sSet$. We say $x \in X_n$ is degenerate if there exist

- $k < n$
- $y \in X_k$
- $\alpha : [n] \longrightarrow [k]$

s.t. $x = y \cdot \alpha$.

x really comes from lower dimension

This is consistent with the previous definition:

$\beta : [n] \longrightarrow [m]$ is not injective $\Leftrightarrow$ β factors as $[n] \longrightarrow [k] \longrightarrow [m]$ for some $k < n$
$\Leftrightarrow$ β is degenerate as an n-simplex in Δ^m.

Eilenberg-Zilber Lemma:
Any n-simplex x in $X \in sSet$ can be written uniquely as $x = y \cdot \alpha$ where $\alpha : [n] \longrightarrow [k]$ is surjective and $y \in X_k$ is non-degenerate.

The non-degenerate simplices are the ones that "really matter".

To construct $\pi_n(X, x)$ of $X \in \underline{\mathrm{Top}}$, we had to talk about S^{n-1}.
The corresponding object in $\underline{sSet}$ is ... $f: D^n \to X$ s.t. $f|_{S^{n-1}}$ is constant at x.

Def: The boundary $\partial\Delta^n$ of Δ^n is the simplicial subset given by

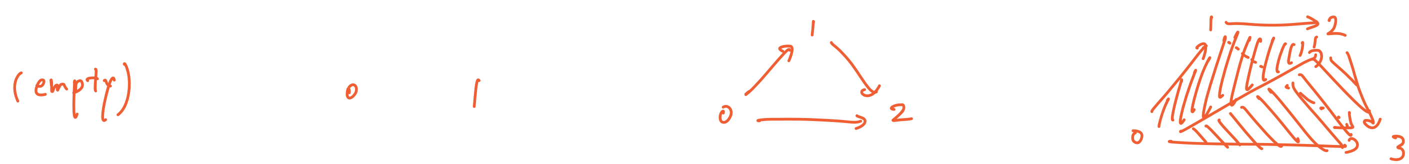
$$(\partial\Delta^n)_m = \{\alpha : [m] \to [n] \text{ in } \Delta \mid \alpha \text{ is NOT surjective}\}.$$

So the non-deg. simplices in $\partial\Delta^n$ are precisely the non-identity injective maps into $[n]$.

e.g.

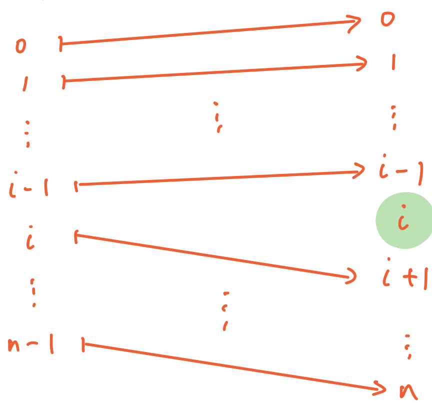
Δ^0	Δ^1	Δ^2	Δ^3

$\partial\Delta^0$	$\partial\Delta^1$	$\partial\Delta^2$	$\partial\Delta^3$



Def: Let $n \geq 1$ and $0 \leq i \leq n$. By the *i-th face* of Δ^n, we mean the map

$$[n-1] \xrightarrow{\delta_i} [n]$$



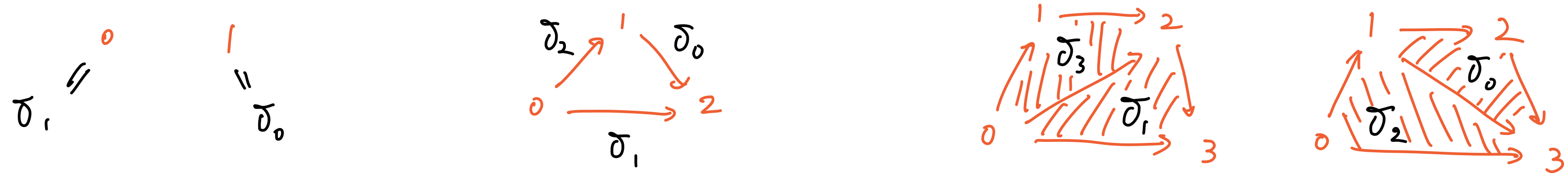
$$\left(s \longmapsto \begin{cases} s & \text{if } s < i \\ s+1 & \text{if } s \geq i \end{cases} \right)$$

face opposite to vertex i

regarded as an $(n-1)$-simplex in Δ^n.

e.g.

$\partial\Delta^1$	$\partial\Delta^2$	$\partial\Delta^3$



In general, the boundary $\partial\Delta^n$ is the union of the faces δ_i.

More precisely, $\alpha : [m] \to [n]$ is in $\partial\Delta^n$ iff it factors through $[n-1] \xrightarrow{\delta_i} [n]$ for some i.

Recall that, in Top, we were only interested in CW-cx. — built by attaching cells

In sSet, everything is CW-complex-like in the following sense.

Def: Let $X \in sSet$ and $d \geq -1$. Then the d-skeleton $sk_d(X)$ of X is the simplicial subset given by

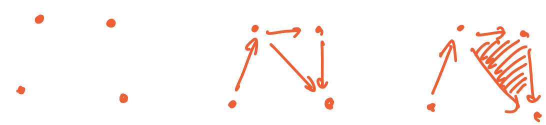
simplices coming from dimension $\leq d$.

$$(sk_d(X))_n = \{ x \in X_n \mid \exists k \leq d \;\; \exists y \in X_k \;\; \exists [n] \xrightarrow{\alpha} [k] \text{ s.t. } x = y \cdot \alpha \}.$$

Convention: $sk_{-1}(X) = \varnothing$.

e.g. $\partial \Delta^n = sk_{n-1}(\Delta^n)$.

(empty)



Fact: Any $X \in sSet$ can be written as $X = \operatorname{colim}\left(sk_{-1}(X) \hookrightarrow sk_0(X) \hookrightarrow sk_1(X) \hookrightarrow sk_2(X) \hookrightarrow \cdots \right)$

Moreover, for each $d \geq 0$, the following square is a pushout:

coproduct over non-deg. d-simplices in X

$$\begin{tikzcd}
\coprod (\partial \Delta^d) \arrow[r] \arrow[d] & sk_{d-1}(X) \arrow[d] \\
\coprod (\Delta^d) \arrow[r] & sk_d(X) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

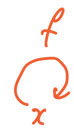
On the other hand, the homotopy groups are harder to define for simplicial sets than for topological spaces.

Def: Let $f, g : X \to Y$ in sSet. Then a homotopy $H : f \sim g$ is a map $H : X \times \Delta^1 \to Y$ s.t. $H(-,0) = f$ & $H(-,1) = g$.

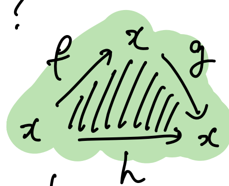
$$X \cong X \times \Delta^0 \xrightarrow{id \times \delta_1} X \times \Delta^1 \xrightarrow{H} Y.$$

Given $X \in$ sSet & $x \in X_0$, we want to define

$$\pi_1(X,x) = \{ f \in X_1 \mid f \cdot \delta_0 = f \cdot \delta_1 = x \} / \text{endpoint-preserving homotopy}.$$

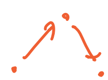
looks like 

But how do we concatenate two loops?

If there is a 2-simplex in X like 

then it seems reasonable to set $g * f = h$, but we don't always have such a 2-simplex.

So we'll restrict our attention to Kan complexes - simplicial sets with "enough simplices".

So that e.g. any  can be completed to 

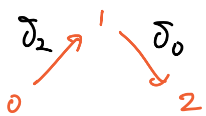
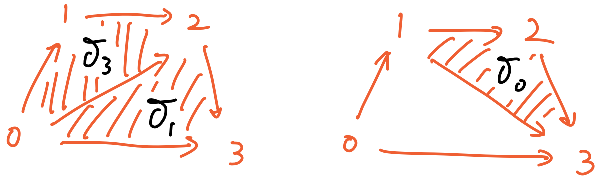
Def: Let $n \geq 1$ and $0 \leq k \leq n$.

The k-th horn Λ^n_k is the union of i-th faces of Δ^n for $i \neq k$.

More precisely,

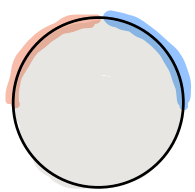
$$(\Lambda^n_k)_m = \{\alpha : [m] \to [n] \text{ in } \Delta \mid \exists\, i \in [n] \text{ s.t. } k \neq i \text{ and } i \notin \mathrm{im}(\alpha)\}.$$

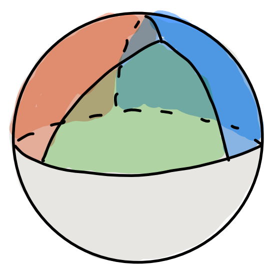
e.g.

Λ^2_1	Λ^3_2
	

The geometric realisation $|-| : \underline{sSet} \to \underline{Top}$ sends the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ to the inclusion of a hemisphere $D^{n-1} \hookrightarrow D^n$.

e.g.





Def: $X \in sSet$ is a Kan complex if any horn in X can be filled to a simplex.

i.e. $\forall n \ \forall k$

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\forall"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$\leftrightsquigarrow$ any $\nearrow\searrow$ can be completed to 

Fact: For any $A \in Top$, $Sing(A)$ is a Kan complex because:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & Sing(A) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} D^{n-1} \cong & {|\Lambda^n_k|} \arrow[r] \arrow[d, hook] & A \\ D^n \cong \arrow[u, dashed] & {|\Delta^n|} \arrow[ur, dashed] & \end{tikzcd}$$

Topologically, we may interpret the horn-filling condition for arbitrary $X \in sSet$ as:
given n many $(n-1)$-simplices in X forming an $(n-1)$-disk,
we can find a single $(n-1)$-simplex representing a disk with the same boundary,
together with a boundary-preserving homotopy connecting the two disks.

i.e. we can paste simplices

Fact: For a Kan complex X, $x \in X_0$ and $n \geq 1$,

$[n-1] \xrightarrow{!} [0]$

$$\pi_n(X,x) = \{ f \in X_n \mid f \cdot \delta_i = x \cdot ! \text{ for each } 0 \leq i \leq n \} / \text{bdry-preserving homotopy}$$

is a group. Moreover, $\pi_n(X,x) \cong \pi_n(|X|, x)$.

Def: A map $f : X \to Y$ in $\underline{\mathrm{sSet}}$ is a weak homotopy equivalence if $|f| : |X| \to |Y|$ is a weak homotopy equivalence in $\underline{\mathrm{Top}}$; equivalently, if

- f induces a bijection between connected components; and
  coequaliser of $X_1 \rightrightarrows X_0$.
- $\pi_n(|f|) : \pi_n(|X|, x) \to \pi_n(|Y|, f(x))$ is an isomorphism for all $n \geq 1$ & all $x \in X_0$.

Fact: Any homotopy equivalence is a weak homotopy equivalence.

Whitehead's theorem:

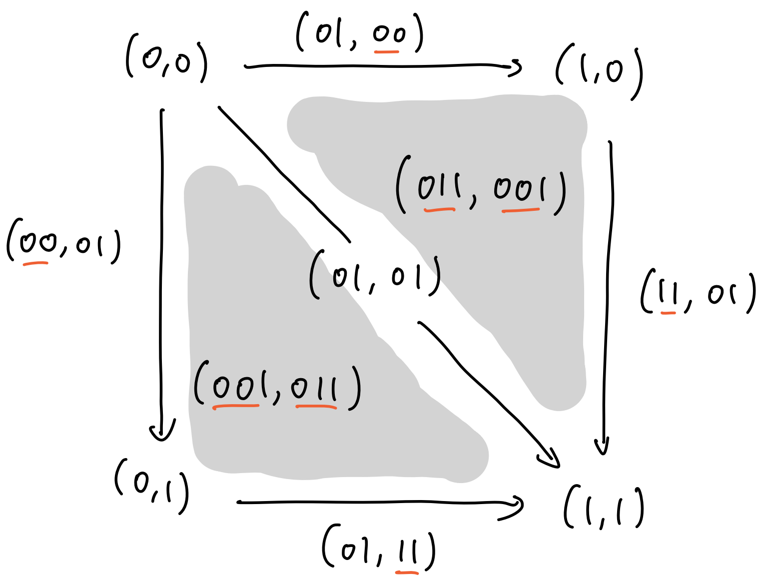
Any weak homotopy equivalence between Kan complexes is a homotopy equivalence.

Next lecture: The adjunction $\underline{\mathrm{Top}} \underset{|\cdot|}{\overset{\mathrm{Sing}}{\underset{\perp}{\rightleftarrows}}} \underline{\mathrm{sSet}}$ exhibits the homotopy theory of CW-complexes & that of Kan complexes to be equivalent.

$\boxed{\text{Extra}}$

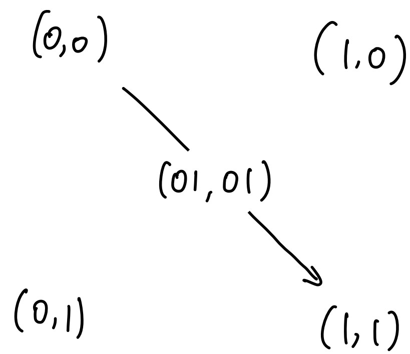
If we only care about non-degenerate simplices, why does Δ have to contain all order-preserving functions rather than just injective ones?

Consider the product $\Delta^1 \times \Delta^1$ (note $(X \times Y)_n = X_n \times Y_n$). Its non-deg. simplices are:



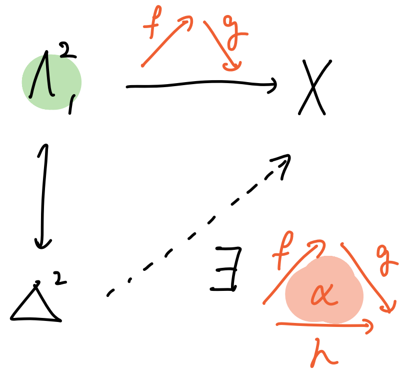
Some of these simplices come from degenerate ones in Δ^1.

If we replace Δ by the injective-only version, the corresponding product looks like:



So we need degenerate simplices to get the "correct" product.

For a Kan complex X & $x \in X_0$, we define the multiplication in $\pi_1(X,x)$ by filling Λ^2_1 : given $f,g : x \to x$,



Observe that, if 

$\in X_2$ then $[h] = [k]$

deg. 1-simplex

since we we can complete it to a homotopy :

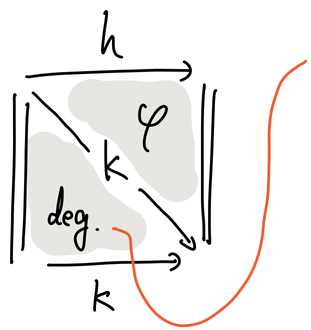


image of $k \in X_1$ under the action of

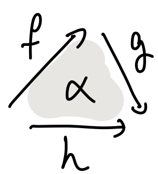
$[2] \to [1]$

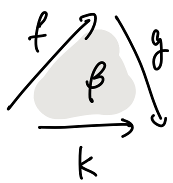
$0 \mapsto 0$

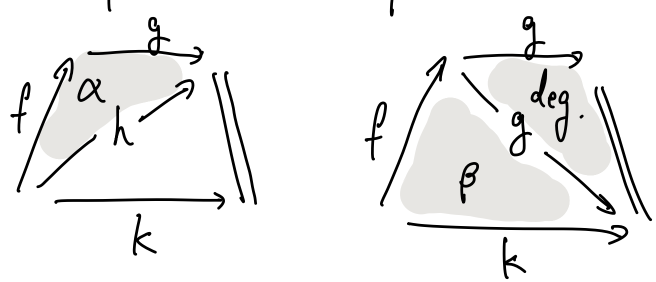
$1 \mapsto 0$

$2 \mapsto 1$

Using this fact & filler for Λ^3_1, we can prove the uniqueness of composite :

given 

& 

, fill 

Exercise : Prove that $\pi_1(X,x)$ is indeed a group.