Lecture 1: Spaces as CW-complexes

By "spaces", we mean CW-complexes up to (weak) homotopy equivalence.

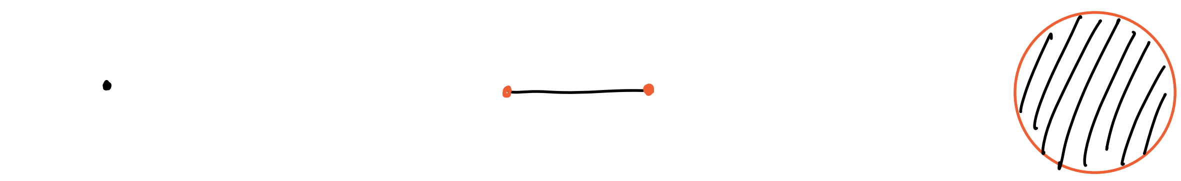
when do we want to think of two spaces as "the same"?

A CW-complex is a topological space built from simple building blocks.

Def: $D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$ disk

$S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 = 1\}$ sphere

e.g.	$S^{-1} = \emptyset \subset D^0$	$S^0 \subset D^1$	$S^1 \subset D^2$



Def: A CW-complex is a topological space X that can be written as

$$X = \operatorname{colim}\left(X^{-1} \to X^0 \to X^1 \to \dots\right)$$

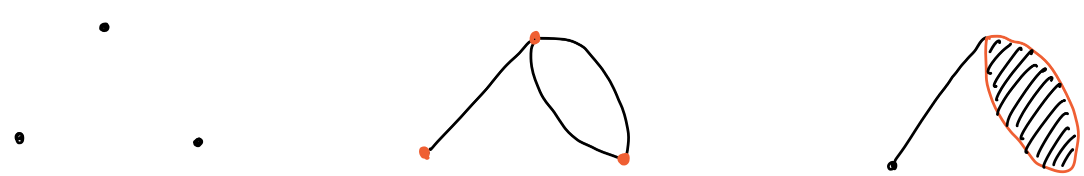
in Top where $X^{-1} = \emptyset$ and each $X^{n-1} \to X^n$ is a pushout of a coproduct of $S^{n-1} \hookrightarrow D^n$:

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\text{p.o.}"] & X^{n-1} \arrow[d] \\ \coprod D^n \arrow[r] & X^n \end{tikzcd}$$

attaching n-cells along boundary

Category of (nice) topological spaces and continuous functions

e.g.	X^0	X^1	X^2



Def: Let $f, g : X \to Y$ in Top.

A homotopy $H : f \sim g$ is a cts map

$$H : X \times I \longrightarrow Y$$

closed unit interval $[0,1]$ with usual topology

s.t. $H(x,0) = f(x)$, $H(x,1) = g(x)$.

H witnesses that f & g are similar as cts maps $X \to Y$

Def: $f : X \to Y$ in Top is a homotopy equivalence if there exist $g : Y \longrightarrow X$ in Top & homotopies $H : gf \sim id_X$, $K : fg \sim id_Y$.

Any homeomorphism is a htpy eq., but there are many more htpy eq.

e.g. $f : \{0\} \hookrightarrow I$ $\quad (K : fg \sim id_I$ given by $K(s,t) = st)$

Homotopy equivalences between CW-cx can be detected by homotopy groups.

Def: Let $X \in \mathrm{Top}$ and $x \in X$.

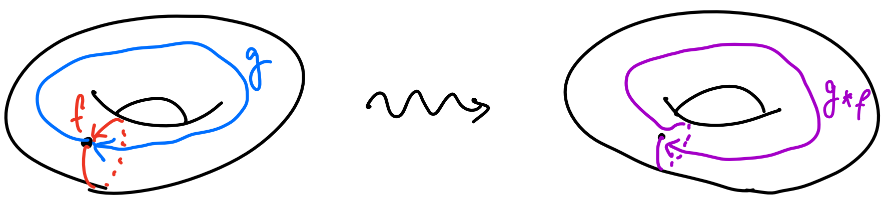
The fundamental group of the pair (X, x) is

$$\pi_1(X,x) = \underbrace{\{\text{loops in } X \text{ based at } x\}}_{\substack{f : I \to X \text{ with} \\ f(0) = f(1) = x}} \Big/ \underbrace{\text{loop homotopy.}}_{\substack{H : I \times I \to X \text{ s.t.} \\ \cdot\ H(-,0) = f \\ \cdot\ H(-,1) = g \\ \cdot\ H(-,t) \text{ is a loop at } x \text{ for all } t.}}$$

H witnesses that f & g are similar as loops in X based at x.

Fact: $\pi_1(X,x)$ is a group.

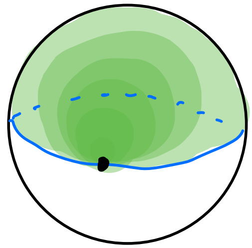
$$\underbrace{[g]}_{\substack{\text{equivalence class} \\ \text{containing } g.}} * [f] = [g * f] \quad \text{where} \quad (g*f)(s) = \begin{cases} f(2s) & \text{if } 0 \le s \le 1/2, \\ g(2s-1) & \text{if } 1/2 \le s \le 1. \end{cases}$$



π_1 counts the number of "1-dimensional" holes, so $\pi_1(S^1) \cong \mathbb{Z}$ but $\pi_1(S^2)$ is trivial.

suppressing the basepoint because any two choices would yield isomorphic groups.

e.g.



To count the higher-dimensional holes, we need higher homotopy groups.

Def: Let $X \in \underline{Top}$, $x \in X$ and $n \geq 1$.

The n-th homotopy group of (X, x) is

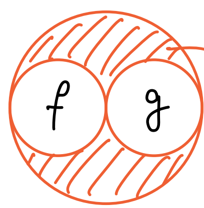
$$\pi_n(X,x) = \left\{ \begin{array}{l} f : D^n \to X \text{ in } \underline{Top} \text{ s.t.} \\ f|_{S^{n-1}} \text{ is constant at } x. \end{array} \right\} \Big/ \text{"boundary-preserving" homotopy}$$

$S^n \to X$

restriction

Fact: $\pi_n(X,x)$ is a group (abelian for $n \geq 2$). We always have $\pi_n(S^n) \cong \mathbb{Z}$.

composition : $g * f =$



sent to x

Facts: • π_n is a functor $\underbrace{Top_*}_{\text{category of } (X,x)\text{'s}} \longrightarrow \underbrace{Grp}_{\text{category of groups}}$.

• If $\exists$ homotopy $f \sim g$ then $\pi_n(f) = \pi_n(g)$

$\rightsquigarrow$ Every homotopy equivalence is a weak homotopy equivalence.

Def: $f: X \to Y$ in Top is a weak homotopy equivalence if

• f induces a bijection between path-components;

$\exists$ path $x \sim x'$ in X $\Leftrightarrow$ $\exists$ path $f(x) \sim f(x')$ in Y

AND

$\forall y \in Y \ \exists x \in X \ \exists$ path $f(x) \sim y$.

• $\pi_n(f): \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism

for all $x \in X$ and for all $n \geq 1$.

if $gf \sim id$ then $\pi_n(g)\, \pi_n(f) = \pi_n(gf) = \pi_n(id) = id$

Whitehead's theorem:

Every weak homotopy equivalence between CW-complexes is a homotopy equivalence.

Today: By "spaces", we mean CW-complexes up to (weak) htpy eq.

But really the "space" corresponding to a CW-cx X just needs to remember the maps $D^n \to X$ (& $S^{n-1} \to X$) and how they fit together.

what we use to construct π_n.

These data can be packaged into a simplicial set $\mathrm{Sing}(X)$. singular complex

Def: The category Δ has

(obj) ordered sets $[n] = \{0 < 1 < \dots < n\}$ for $n \geq 0$; and

(mor) order-preserving functions between them.

$i \leq j \Rightarrow f(i) \leq f(j)$

The category of simplicial sets is the functor category

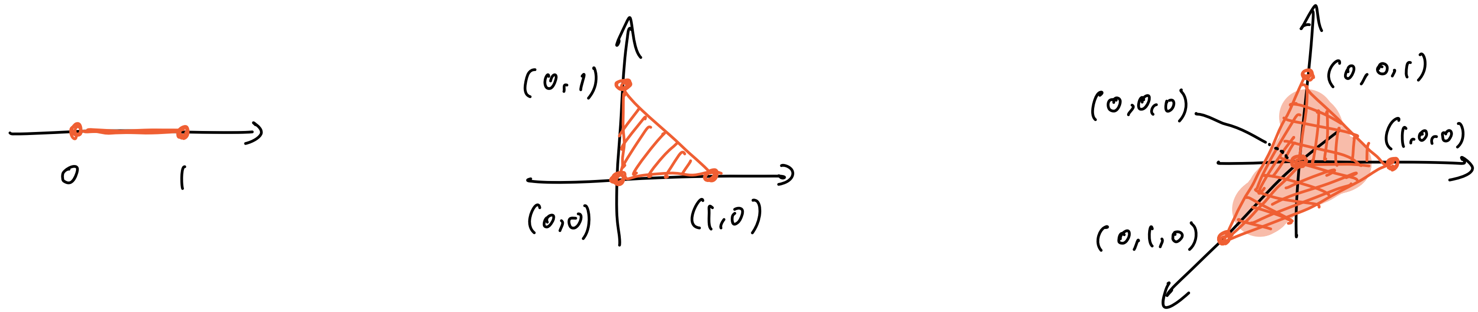
$\underline{sSet} = [\Delta^{op}, \underline{Set}]$.

Def: For $n \geq 0$, the topological n-simplex $|\Delta^n|$ is the convex hull in $\mathbb{R}^n$ of

$(0,\dots,0)$, $(1,0,\dots,0)$, $(0,1,0,\dots,0)$, $\dots$, $(0,\dots,0,1)$.
"0" "1" "2" "n"

e.g.

$\lvert\Delta^1\rvert$	$\lvert\Delta^2\rvert$	$\lvert\Delta^3\rvert$



We can extend $[n] \longmapsto |\Delta^n|$ to a functor $\Delta \longrightarrow \mathrm{Top}$

$f : [m] \to [n] \rightsquigarrow f_* : |\Delta^m| \to |\Delta^n|$

linear extension of action on vertices

Def: The singular complex functor is given by

$$\mathrm{Sing} : \underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{sSet}} = [\Delta^{op}, \underline{\mathrm{Set}}]$$
$$X \longmapsto \big([n] \mapsto \underline{\mathrm{Top}}(|\Delta^n|, X)\big).$$

Fact: Sing has a left adjoint $|-| : \underline{sSet} \to \underline{Top}$ which sends the representable

$$\Delta^n = \underline{sSet}(-, [n])$$

to the topological n-simplex $|\Delta^n|$.

geometric realisation

$Sing(X)$ indeed remembers all $D^n \to X$ & $S^{n-1} \to X$ since $|\Delta^n| \cong D^n$ and $|\partial\Delta^n| \cong S^{n-1}$ in $\underline{Top}$:

$$D^n \to X \quad \leftrightsquigarrow \quad |\Delta^n| \to X \quad \leftrightsquigarrow \quad \Delta^n \to Sing(X)$$

$$S^{n-1} \to X \quad \leftrightsquigarrow \quad |\partial\Delta^n| \to X \quad \leftrightsquigarrow \quad \partial\Delta^n \to Sing(X)$$

in $\underline{Top}$ in $\underline{sSet}$

we'll see $\partial\Delta^n$ in next lecture.

Next lecture: Forget about topological spaces and do homotopy theory with simplicial sets.

In general, any functor $F : \Delta \to \mathcal{C}$ induces an adjunction

cocomplete

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "R"] \end{tikzcd}$$

where $(RX)([n]) = \mathcal{C}(F[n], X)$ and L is an (essentially unique) cocontinuous extension of F.

Look up "free cocompletion". This actually works with any small category in place of Δ.

We'll think of $[n] = \{0 < 1 < \dots < n\}$ as a combinatorial representation of the topological n-simplex $|\Delta^n|$ but "directed" i.e. equipped with ordering on vertices

e.g.

[0]	[1]	[2]	[3]

So F is specifying what we mean by "simplices in $\mathcal{C}$", & $(RX)([n])$ is the set of copies of n-simplex we can find inside X.

For example, we can obtain $F: \Delta \to \underline{Cat}$ by regarding each $[n] = \{0 < 1 < \dots < n\}$ as a category. ($i \leq j \iff$ there is a unique morphism $i \to j$)

The right adjoint $N: \underline{Cat} \to \underline{sSet}$ of the induced adjunction is called the nerve functor.

An element of $(N\mathcal{C})([n])$ is a commutative n-simplex in $\mathcal{C}$ (which amounts to a composable sequence of n morphisms).

Now we have seen simplicial sets coming from topological spaces & categories. The notion of ∞-category (or more accurately quasi-category) we'll eventually get to subsumes both of these.