Equivalences in and between algebraic weak ω-categories

j/w Soichiro Fujii[1] and Keisuke Hoshino

Yuki Maehara[2]

Kyoto University

(∞, n)-Categories and their applications

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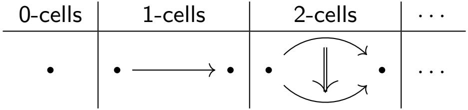
[2]Supported by JSPS KAKENHI Grant Number JP21K20329 & JP23K12960

1. Algebraic weak ω-categories
2. Equivalences *in* an algebraic weak ω-category
3. (Weak) equivalences *between* algebraic weak ω-categories

Globular approach



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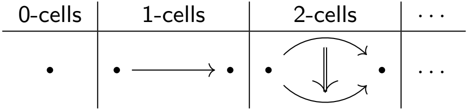

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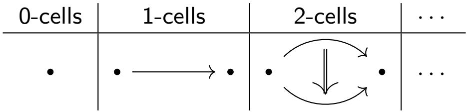

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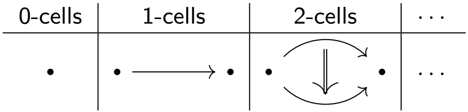

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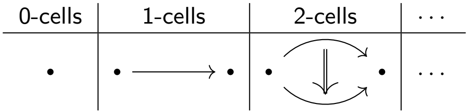

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How should we define T_{wk}?

We should have {strict ω-cats} $\subset$ {weak ω-cats}, or equivalently a monad map $\alpha : T_{\mathrm{wk}} \to T_{\mathrm{st}}$.

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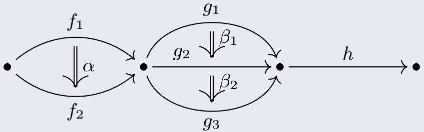

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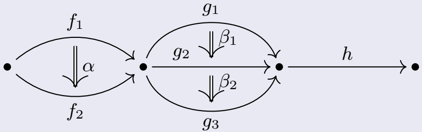

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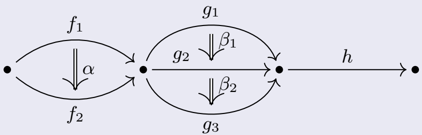

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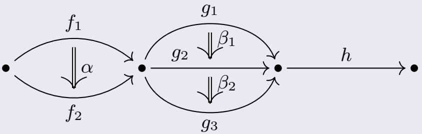

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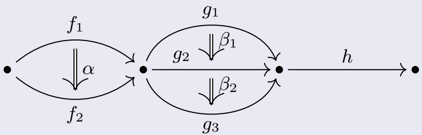

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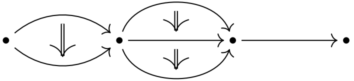
- $(T_{\mathrm{st}}1)_2$ contains cells like  .

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Existence part of Pasting Theorem

We ask that each commutative square

$$\begin{array}{ccc} \partial G^n & \longrightarrow & T_{\mathrm{wk}}1 \\ \downarrow & \nearrow & \downarrow{\scriptstyle \alpha_1} \\ G^n & \longrightarrow & T_{\mathrm{st}}1 \end{array}$$

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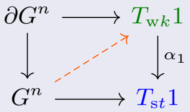

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The data of such lifts is called a contraction.

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Definition (Leinster)

T_{wk} is the initial cartesian monad over T_{st} with contraction.

Identity and binary composition

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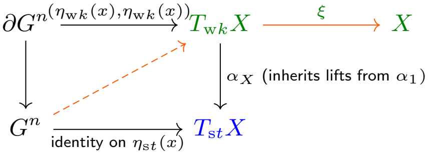

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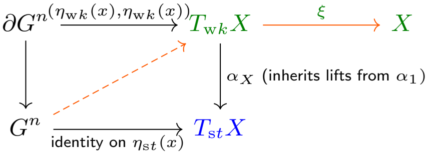

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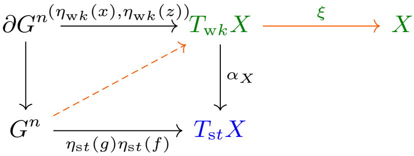

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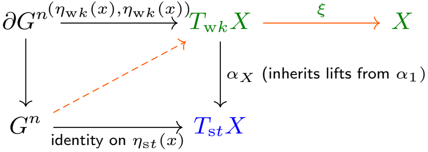

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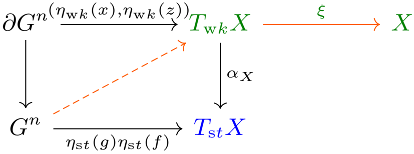

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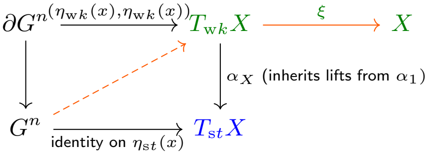

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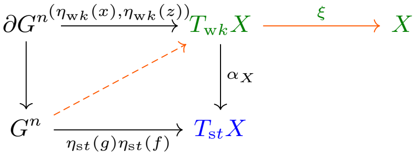

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"f is an equivalence" means "f admits such an infinite hierarchy of witnesses"

Uniqueness and applications

Uniqueness part of Pasting Theorem

Let $(X, T_{\mathrm{wk}}X \xrightarrow{\xi} X)$ *be a weak* ω*-category. If* $f /\!\!/ g$ *in* $(T_{\mathrm{wk}}X)_n$ *and* $\alpha_X(f) = \alpha_X(g)$ *then there is an equivalence* $(n+1)$*-cell* $\xi(f) \to \xi(g)$ *in* X*.*

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$$h(gf) \sim (hg)f, \quad 1f \sim f \sim f1 \quad \text{etc.}$$

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The class of equivalence n*-cells in a weak* ω*-category is closed under pastings.*

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Using these theorems, we can treat weak ω-categories just like strict ones.

Uniqueness and applications

Uniqueness part of Pasting Theorem

Let $(X, T_{\mathrm{wk}}X \xrightarrow{\xi} X)$ be a weak ω-category. If $f /\!/ g$ in $(T_{\mathrm{wk}}X)_n$ and $\alpha_X(f) = \alpha_X(g)$ then there is an equivalence $(n+1)$-cell $\xi(f) \to \xi(g)$ in X.

Instances of this result yield:

$$h(gf) \sim (hg)f, \quad 1f \sim f \sim f1 \quad \text{etc.}$$

For more non-trivial things, we need:

Theorem

The class of equivalence n-cells in a weak ω-category is closed under pastings.

Proof

Requires more work than one might expect because one has to deal with both "formal composites" and "actual composites."

Using these theorems, we can treat weak ω-categories just like strict ones...?

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A weak equivalence $F : X \to Y$ is a $T_{\mathrm{w}k}$-algebra morphism such that

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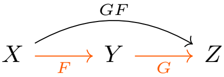
Theorem

The class of weak equivalences enjoys the 2-out-of-3 property.
That is, if any two of F, G and GF are weak equivalences then so is the third.

Case 1: F and G are weak equivalences



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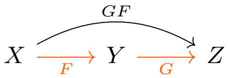

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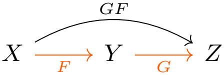

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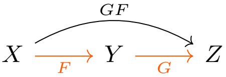

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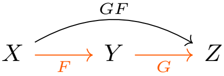

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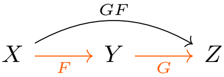

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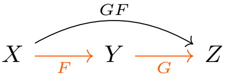

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Let $x, x' \in X_0$. Then we have

$$X(x,x') \xrightarrow{F_{x,x'}} Y(Fx,Fx') \xrightarrow{G_{Fx,Fx'}} Z(GFx,GFx')$$
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and we can repeat the argument above.

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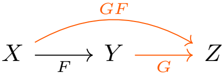

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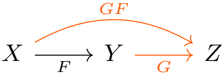

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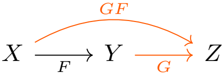

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$$X \xrightarrow[F]{} Y \xrightarrow[G]{} Z, \quad X \xrightarrow{GF} Z$$

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Let $x, x' \in X_0$ and consider

$$X(x,x') \xrightarrow[F_{x,x'}]{} Y(Fx,Fx') \xrightarrow[G_{Fx,Fx'}]{} Z(GFx,GFx'), \quad X(x,x') \xrightarrow{(GF)_{x,x'}} Z(GFx,GFx')$$

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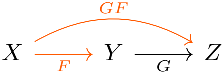

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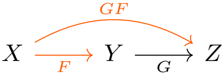

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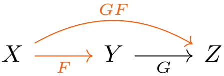

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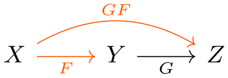

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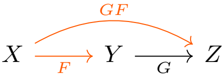

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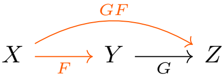

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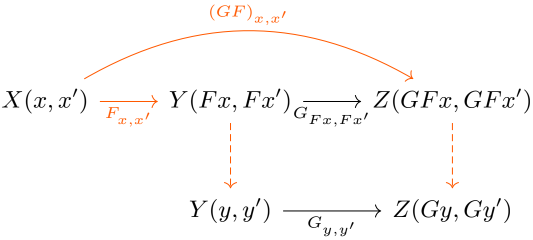

We need whiskering!

Whiskering

We want:

Lemma

For an equivalence 1*-cell* $u \colon y \to z$ *in a weak* ω*-category* X*, the whiskering map*

$$u * (-) \colon X(x, y) \to X(x, z)$$

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The proof of the strict case (Lafont-Métayer-Worytkiewicz) implicitly relies on:

Obvious fact in strict case

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is (the identity and so in particular) a weak equivalence.

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$$\begin{aligned} & u * (-) \text{ is essentially injective} \\ \implies 1_y * (-) \text{ is essentially surjective} \implies{} & u * (-) \text{ is essentially surjective.} \end{aligned}$$

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Thank you!

Papers (Fujii-Hoshino-M.)

- *Weakly invertible cells in a weak ω-category*, to appear in Higher Structures, arXiv:2303.14907
- *ω-weak equivalences between weak ω-categories*, will put up on arXiv soon
- more to come!