Introduction to ∞-categories in general and quasi-categories in particular

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Thursday Seminar, Kyoto

1. What is an ∞-category?
2. Quasi-categories via spines
3. Quasi-categories via inner horns
4. Conclusion

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It seems "obvious" that ∞-categories should just be $\underline{\mathrm{Top}}$-categories, but...

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H : "$A \times I$" $\to B$ with $H(-,0) = f$ and $H(-,1) = g$.

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(These different models have been shown to be equivalent to each other in a suitable sense.)

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- $(N\mathscr{A})_2$ = commutative triangles in $\mathscr{A}$

Quasi-categories, intuitively

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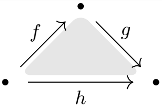
e.g. 2-simplex 

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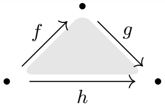
e.g. 2-simplex 

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The precise definition may be obtained by "homotopifying" a characterisation of $N\mathscr{A}$'s in $\underline{\mathrm{sSet}}$.

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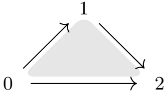
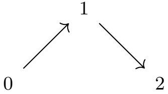
$n=2$		

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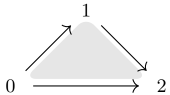
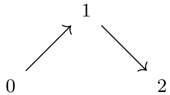
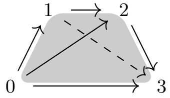
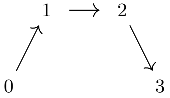
$n = 2$		
$n = 3$		

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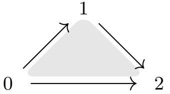
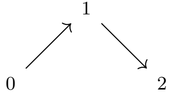
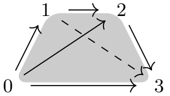
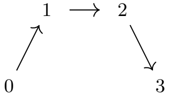
$n = 2$		
$n = 3$		

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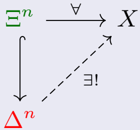

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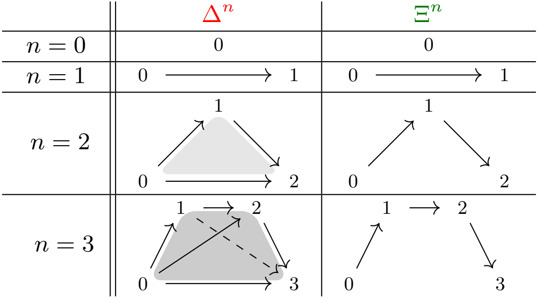

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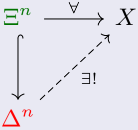

Quasi-categories via spines

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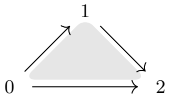
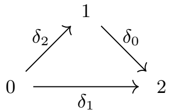
$n = 2$		

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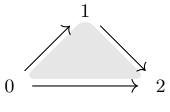
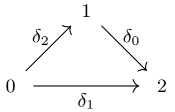
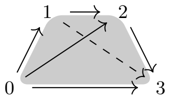
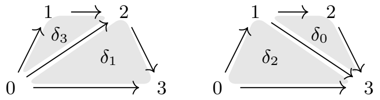
$n=2$			
$n=3$			

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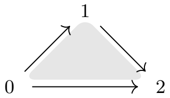
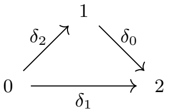
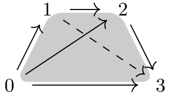
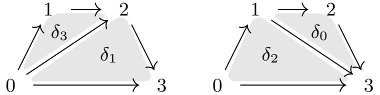
$n = 2$			
$n = 3$			

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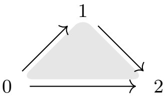
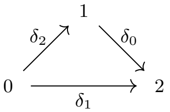
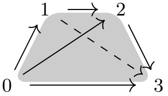
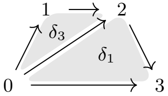
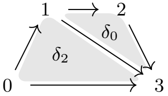
$n = 2$			
$n = 3$			

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Quasi-categories via inner horns

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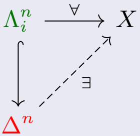

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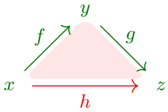

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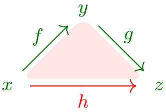

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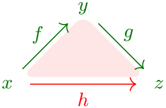

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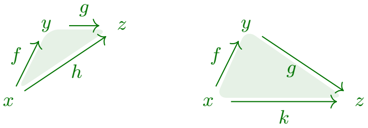

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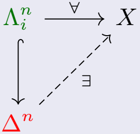

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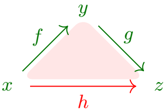

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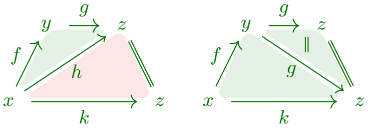

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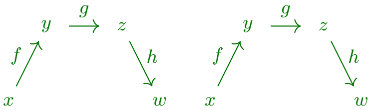

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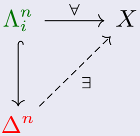

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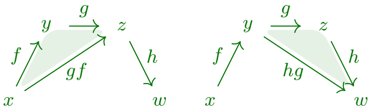

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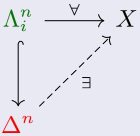

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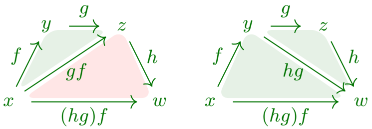

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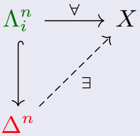

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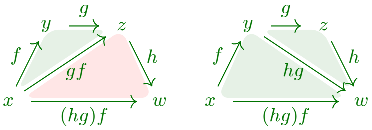

- Higher-dimensional horns encode "higher coherence".

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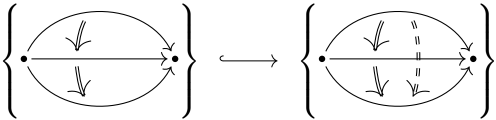
In practice

Alternative definition of quasi-category using inner horns is much more popular.

My own work

- Developed inner horns for 2-quasi-categories.



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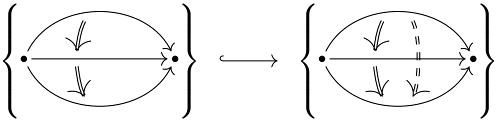

- Used those horns to analyse Gray tensor product for 2-quasi-categories.

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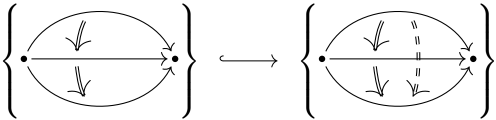
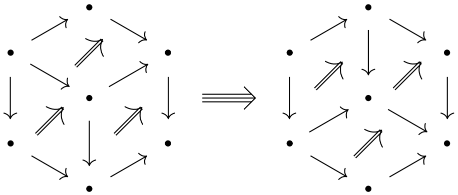


- Used those horns to analyse Gray tensor product for 2-quasi-categories.
- Proposed a cubical model for (∞, ∞)-categories.
  (j/w Tim Campion and Chris Kapulkin)



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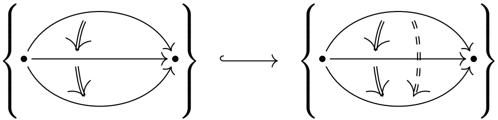

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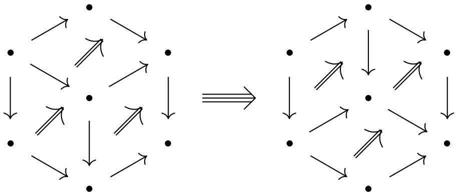

- Proved freeness of certain simplicial (∞, ∞)-categories.