Coinductive equivalences in algebraic weak ω-categories

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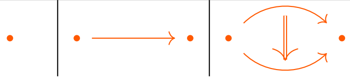
[1] Supported by JSPS KAKENHI Grant Number JP21K20329

0-cells	1-cells	2-cells	$\cdots$



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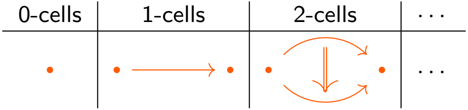

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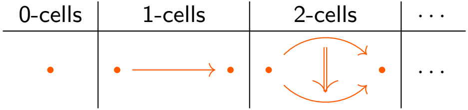

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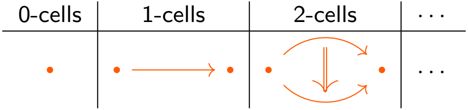

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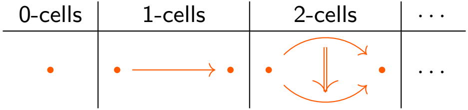

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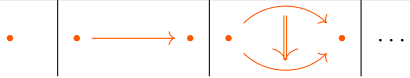

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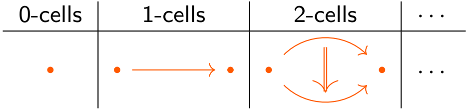

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We should have {strict ω-cats} $\subset$ {weak ω-cats}, or equivalently a monad map $\alpha : T^{\mathrm{w}k} \to T^{\mathrm{st}}$.

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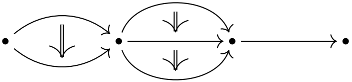
- $(T^{st}1)_2$ contains cells like  ,

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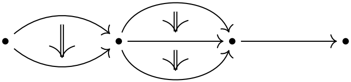
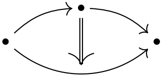
- $(T^{\mathrm{st}}1)_2$ contains cells like  ,

but not 

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Existence part of Pasting Theorem

We ask that any commutative square

$$\begin{array}{ccc} \partial G^n & \longrightarrow & T^{\mathrm{wk}}1 \\ \downarrow & \nearrow & \downarrow{\scriptstyle \alpha_1} \\ G^n & \longrightarrow & T^{\mathrm{st}}1 \end{array}$$

admit a chosen diagonal lift for $n \geq 1$.

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By a *weak ω-category*, we mean a T^{wk}-algebra.

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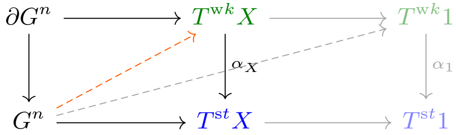

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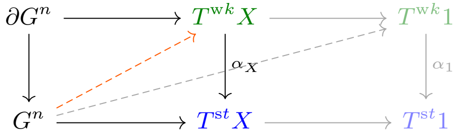

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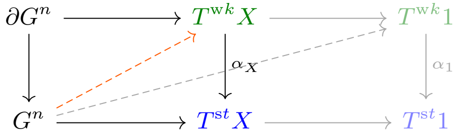

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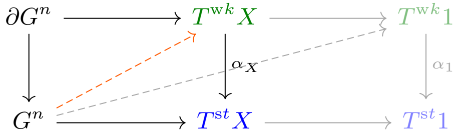

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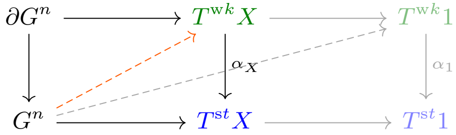

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"f is an equivalence" means "f admits such an infinite hierarchy of witnesses"

Uniqueness and applications

Uniqueness part of Pasting Theorem

Let $(X, T^{\mathrm{wk}}X \xrightarrow{\xi} X)$ *be a weak* ω*-category. If* $f \mathbin{/\!/} g$ *in* $(T^{\mathrm{wk}}X)_n$ *and* $\alpha_X(f) = \alpha_X(g)$ *then there is an equivalence* $(n+1)$*-cell* $\xi(f) \to \xi(g)$ *in* X*.*

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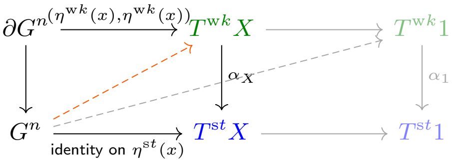
That's it!

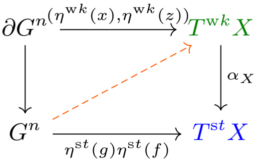
Thank you!

Identity and binary composition

Let $(X, T^{\mathrm{wk}}X \xrightarrow{\xi} X)$ be a weak ω-category and $x \in X_{n-1}$.
We can define $1_x \in X_n$ by applying ξ to the lift in



Similarly, given n-cells $x \xrightarrow{f} y \xrightarrow{g} z$, we can define $gf \in X_n$ using



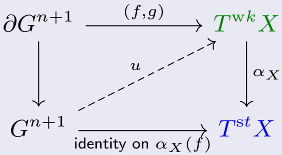
Uniqueness

Uniqueness part of Pasting Theorem

Let $(X, T^{\mathrm{wk}}X \xrightarrow{\xi} X)$ *be a weak* ω*-category. If* $f /\!/ g$ *in* $(T^{\mathrm{wk}}X)_n$ *and* $\alpha_X(f) = \alpha_X(g)$ *then there is an equivalence* $(n+1)$*-cell* $\xi(f) \to \xi(g)$ *in* X*.*

Proof.

We proceed by coinduction. Obtain $u : f \to g$ as



and similarly $v : g \to f$. Then we have $vu /\!/ 1_f$ and $uv /\!/ 1_g$ in $(T^{\mathrm{wk}}X)_{n+1}$, and $\alpha_X(vu) = 1_{\alpha_X(f)} = \alpha_X(1_f)$ and $\alpha_X(uv) = 1_{\alpha_X(g)} = \alpha_X(1_g)$. □