The Gray tensor product for 2-quasi-categories

Yuki Maehara

Macquarie University / Kyushu University

AustMS 2020

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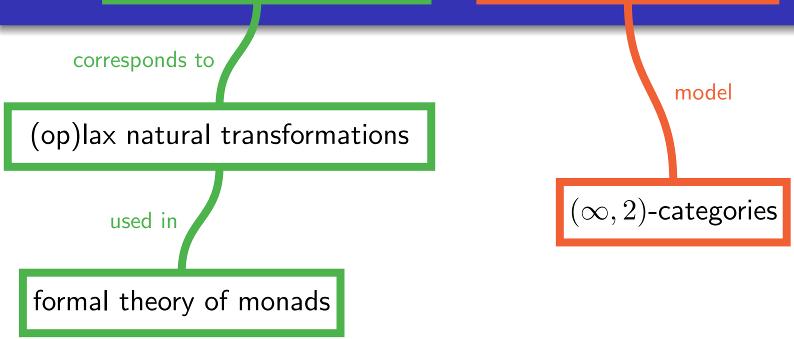

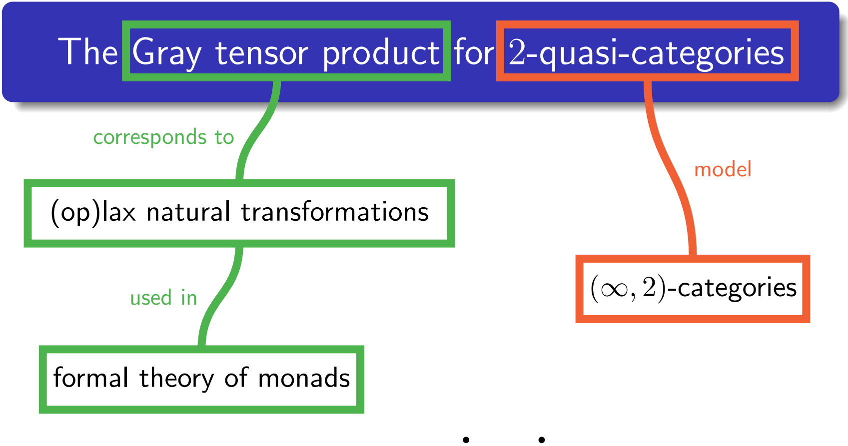
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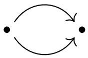

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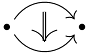

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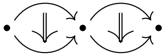

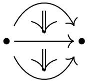

Gray tensor product of 2-categories

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cartesian product $\mathscr{A} \times \mathscr{B}$

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In cartesian product $\mathscr{A} \times \mathscr{B}$

$$\left.\begin{array}{l} x \xrightarrow{\quad f \quad} x' \text{ in } \mathscr{A} \\[2ex] y \xrightarrow{\quad g \quad} y' \text{ in } \mathscr{B} \end{array}\right\}$$

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commutes.

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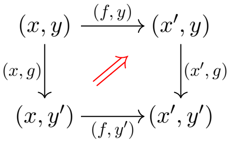

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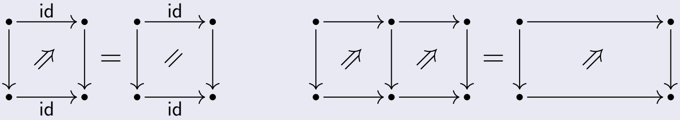
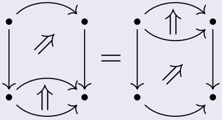
Coherence conditions





etc.

quasi-categories

nerves of categories $\xrightarrow{\text{homotopify}}$ quasi-categories

(2-)quasi-categories

nerves of (2-)categories $\xrightarrow{\text{homotopify}}$ (2-)quasi-categories

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Δ = category of free categories $[n]$

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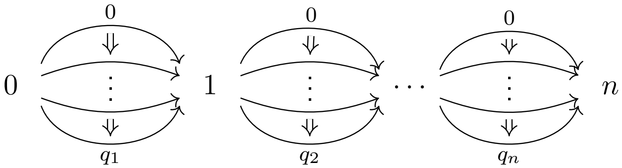
Θ_2 = category of free 2-categories $[n; q_1, \ldots, q_n]$



2-quasi-categories

Definition (Ara)

2-quasi-category = fibrant object in $\widehat{\Theta_2}$

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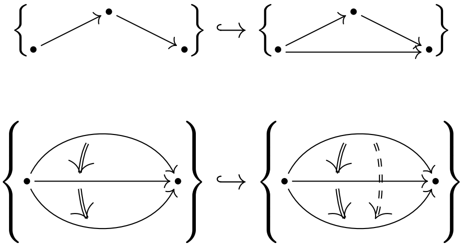
Characterisation (AustMS 2018)

via Oury's inner horn inclusions and equivalence extensions



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Theorem (M.)

The resulting bifunctor $\widehat{\Theta_2} \times \widehat{\Theta_2} \xrightarrow{\otimes} \widehat{\Theta_2}$ *is left Quillen.*

Tensoring three arrows

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Ordinary Gray tensor product $\nearrow \boxtimes \searrow \boxtimes \downarrow$ looks like:

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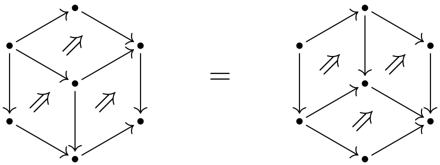

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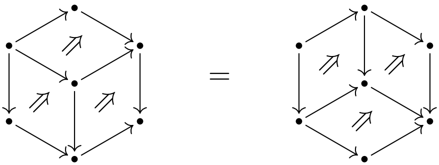

How about 2-quasi-categorical one?

Tensoring three arrows

Focusing on “front” half...

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$$\left(\nearrow \otimes \searrow\right) \otimes \downarrow$$

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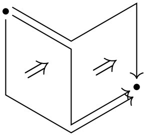

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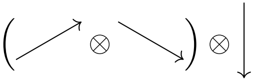
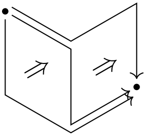
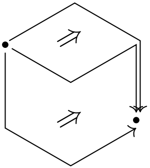
 contains  and 

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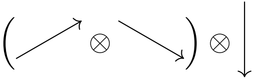

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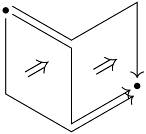

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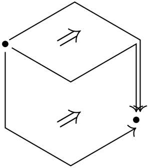

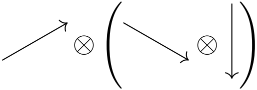

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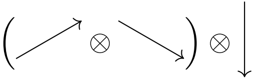

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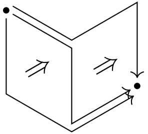

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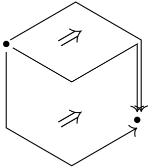

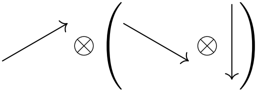

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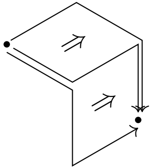

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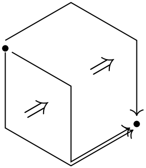

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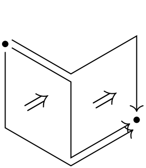
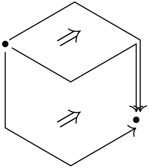
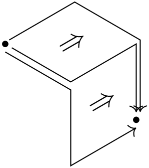
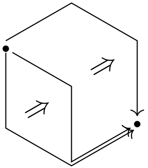
$\left(\nearrow \otimes \searrow\right) \otimes \downarrow$ contains  and 

$\nearrow \otimes \left(\searrow \otimes \downarrow\right)$ contains  and 

Observation

$\otimes$ is NOT associative

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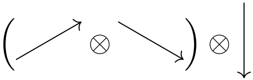
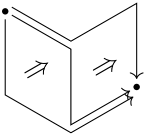
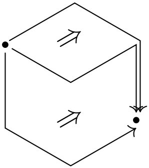
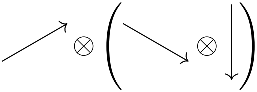
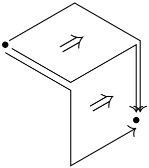
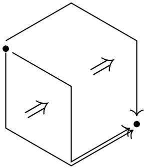
 contains  and 

 contains  and 

Observation

$\otimes$ is NOT associative up to isomorphism

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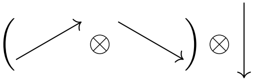
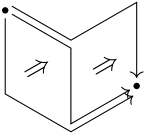
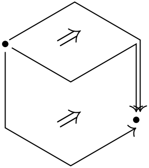
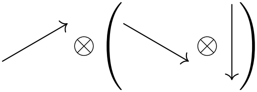
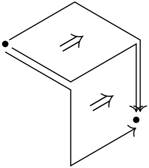
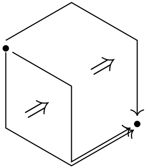
 contains  and 

 contains  and 

Observation

$\otimes$ is NOT associative up to isomorphism
...but maybe up to homotopy?

n-ary tensor

Definition

Define n-ary Gray tensor product by extending

$$\underbrace{\Theta_2 \times \cdots \times \Theta_2}_{n} \hookrightarrow \underbrace{2\text{-}\underline{\mathrm{Cat}} \times \cdots \times 2\text{-}\underline{\mathrm{Cat}}}_{n} \xrightarrow{\boxtimes_n} 2\text{-}\underline{\mathrm{Cat}} \xrightarrow{\text{nerve}} \widehat{\Theta_2}$$

cocontinuously in each variable.

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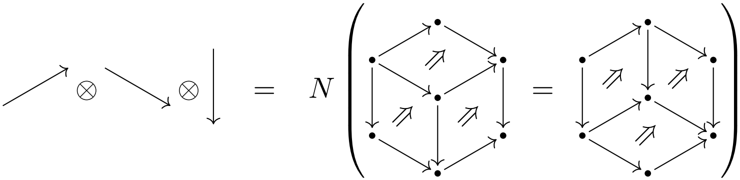
So that:



Associativity up to homotopy

Proposition

These form a lax monoidal structure on $\widehat{\Theta_2}$.

Associativity up to homotopy

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These form a *lax monoidal structure* *on* $\widehat{\Theta_2}$.

e.g. We have comparison maps

$$\otimes_2(\otimes_2(X, Y), Z) \to \otimes_3(X, Y, Z)$$

Associativity up to homotopy

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Theorem (M.)

(The relative version of) these comparison maps are trivial cofibrations.

That's it!

Thank you!