Inner horns for 2-quasi-categories

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Δ

Δ consists of free categories $[n]$:

$$0 \longrightarrow 1 \longrightarrow \dots \longrightarrow n$$

Δ and Θ_2

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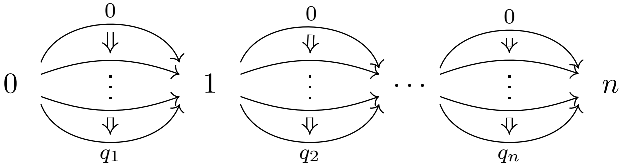
Θ_2 consists of free 2-categories $[n;\mathbf{q}] = [n; q_1, \dots, q_n]$:



Main theorem

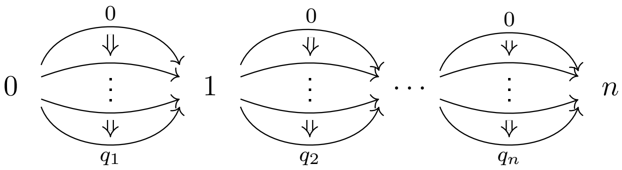
Theorem

$f : X \to Y$ into fibrant Y is a fibration in Ara's model structure iff it has RLP wrt

- *vertical inner horn inclusions;*
- *horizontal inner horn inclusions;*
- *vertical equivalence extensions; and*
- *horizontal equivalence extensions.*

Θ_2

An object $[n; \mathbf{q}] \in \Theta_2$



can be thought of as $[n] \in \Delta$ with labels $[q_i] \in \Delta$:

$$0 \xrightarrow{[q_1]} 1 \xrightarrow{[q_2]} \cdots \xrightarrow{[q_n]} n$$

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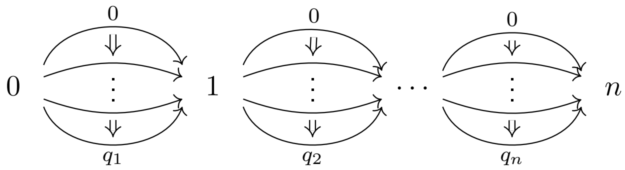

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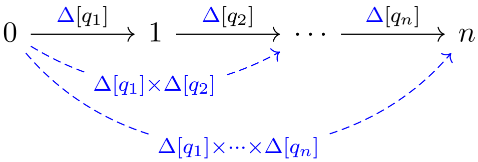

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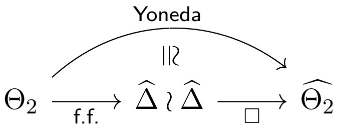

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$$\Theta_2 \xrightarrow[\text{f.f.}]{} \widehat{\Delta} \wr \widehat{\Delta} \xrightarrow[\square]{} \widehat{\Theta_2}, \qquad \Theta_2 \xrightarrow{\text{Yoneda}} \widehat{\Theta_2}, \quad \cong$$

analogous to

$$\Delta \times \Delta \xrightarrow[\text{f.f.}]{} \widehat{\Delta} \times \widehat{\Delta} \xrightarrow[\square]{} \widehat{\Delta \times \Delta}, \qquad \Delta \times \Delta \xrightarrow{\text{Yoneda}} \widehat{\Delta \times \Delta}, \quad \cong$$

Oury's elementary anodyne extensions

$$\widehat{\Delta}/\Delta[n] \times \underbrace{\widehat{\Delta} \times \cdots \times \widehat{\Delta}}_{n} \longrightarrow \widehat{\Delta} \wr \widehat{\Delta}$$

Oury's elementary anodyne extensions

Oury uses

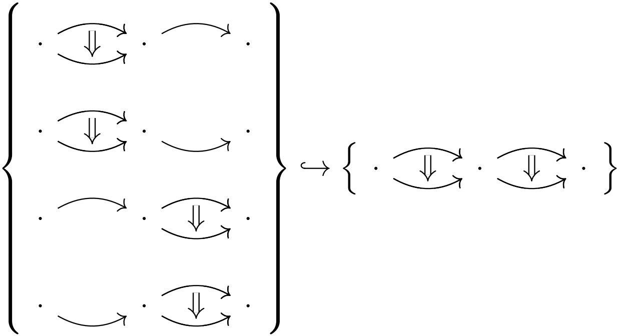
$$\square_n : \widehat{\Delta}/\Delta[n] \times \underbrace{\widehat{\Delta} \times \cdots \times \widehat{\Delta}}_{n} \longrightarrow \widehat{\Delta} \wr \widehat{\Delta} \xrightarrow{\square} \widehat{\Theta_2}$$

to combine

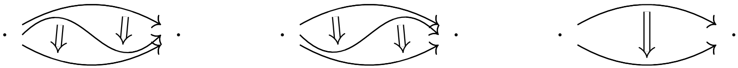
- boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$;
- inner horn inclusions $\Lambda^k[n] \hookrightarrow \Delta[n]$; and
- equivalence extension $\{\cdot\} \hookrightarrow \{\cdot \cong \cdot\}$.

Example : $\Lambda^1_h[2;1,1]$

$\Lambda^1_h[2;1,1] \hookrightarrow \Theta_2[2;1,1]$ looks like:



Missing faces:



Main theorem (again)

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Applications

- lax Gray tensor product is left Quillen;
- $(A \times \{\cdot \to \cdot\}) \cup (A_0 \times \{\cdot \cong \cdot\})$ is a cylinder object for $A \in \widehat{\Theta_2}$;
- special outer horn inclusions are trivial cofibrations;
- htpy coherent nerve of $\mathbf{qCat}$-enriched category is a 2-quasi-category;
- any map (simplicial computad $\to$ htpy coherent nerve) can be strictified to (simplicial computad $\to$ strict nerve).

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Thank you for listening!