Mahavier Limits

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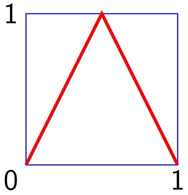
Example: the Bucket-handle Continuum

Let $f\colon I \to I$ be given by



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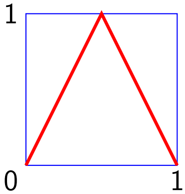
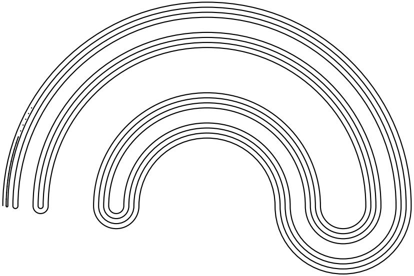
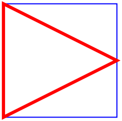
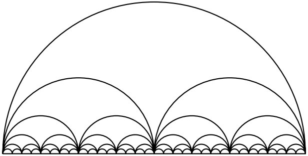


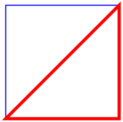
then $\mathrm{Lim}(\ I \xleftarrow{f} I \xleftarrow{f} I \leftarrow \dots\ )$ looks like

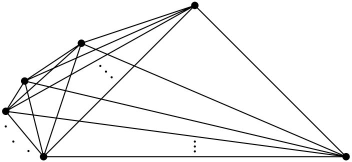


Examples of Mahavier limits









Definition of ordinary limit

Given a sequence of continuous functions between compact Hausdorff spaces

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \longleftarrow \dots$$

its limit can be constructed as

$$\mathrm{Lim}\langle \boldsymbol{X}, \boldsymbol{f} \rangle = \left\{ (x_n)_{n\in\omega} \in \prod_{n\in\omega} X_n : \forall n[x_n = f_{n+1}(x_{n+1})] \right\}.$$

Definition of Mahavier limit

Given a sequence of upper semi-continuous, multi-valued functions between compact Hausdorff spaces

$$X_0 \overset{f_1}{\leftsquigarrow} X_1 \overset{f_2}{\leftsquigarrow} X_2 \leftsquigarrow \dots$$

its Mahavier limit is

$$\mathrm{MahLim}\langle \boldsymbol{X}, \boldsymbol{f} \rangle = \left\{ (x_n)_{n\in\omega} \in \prod_{n\in\omega} X_n : \forall n [x_n \in f_{n+1}(x_{n+1})] \right\}.$$

The (2-)category **CompHausMult**

Definition

The category **CompHausMult** has

- objects: compact Hausdorff spaces
- morphisms: upper semi-continuous functions
- partial order on hom-sets: point-wise inclusion.

SCON's and Mahavier limits

The canonical projections form a single-valued-component oplax natural transformation (SCON).

$$
\begin{array}{ccccccc}
\mathrm{MahLim}\langle \boldsymbol{X}, \boldsymbol{f} \rangle & = & \mathrm{MahLim}\langle \boldsymbol{X}, \boldsymbol{f} \rangle & = & \mathrm{MahLim}\langle \boldsymbol{X}, \boldsymbol{f} \rangle & = & \dots \\
\downarrow \pi_0 & \subseteq & \downarrow \pi_1 & \subseteq & \downarrow \pi_2 & & \\
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\end{array}
$$

Proposition

There is an adjunction

$$
\mathbf{CompHaus} \; \underset{\mathrm{MahLim}}{\overset{\textit{constant}}{\rightleftarrows}} \;\bot\; \mathbf{SCON}(\omega^{\mathrm{op}}, \mathbf{CompHausMult})
$$

Mahavier limits as enriched weighted limits

Mahavier limits should be $\mathscr{V}$-enriched weighted limits where...

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The category $\mathscr{V}$ has

- objects: pairs (A, A') consisting of a poset A and a subset $A' \subseteq A$
- morphisms: order preserving functions $f\colon A \to B$ satisfying $f(A') \subseteq B'$.

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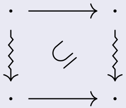
WRONG!!!!!

Mahavier limits as double limits

Definition

The double category $\mathbb{C}\mathbf{ompHaus}$ has

- objects: compact Hausdorff spaces
- horizontal arrows: continuous functions
- vertical arrows: upper semi-continuous functions
- squares: 

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Then $\mathbf{SCON}(\omega^{\mathrm{op}}, \mathbf{CompHausMult})$ is the horizontal part of the functor double category $[\mathbb{V}\omega^{\mathrm{op}}, \mathbb{C}\mathbf{ompHaus}]$.

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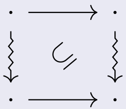
- squares: 

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Proposition

The Mahavier limit of $\langle \boldsymbol{X}, \boldsymbol{f} \rangle$ is the regular (horizontal) double limit of the corresponding double functor $\mathbb{V}\omega^{\mathrm{op}} \to \mathbb{C}\mathbf{ompHaus}$.

Mahavier limits as double limits

Corollary

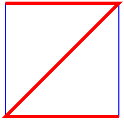
The adjunction of ordinary categories

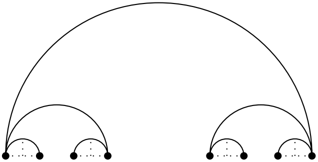
$$\mathbf{CompHaus} \underset{\text{MahLim}}{\overset{\textit{constant}}{\rightleftarrows}} \mathbf{SCON}(\omega^{\text{op}}, \mathbf{CompHausMult}) \quad (\bot)$$

can be upgraded to an adjunction of double categories (of strict/lax type)

$$\mathbb{C}\mathbf{ompHaus} \underset{\text{MahLim}}{\overset{\textit{diagonal}}{\rightleftarrows}} [\mathbb{V}\omega^{\text{op}}, \mathbb{C}\mathbf{ompHaus}] \quad (\bot)$$

Thank you!





Thank you for listening!