Equivalence of cubical and simplicial approaches to weak ω-categories

Yuki Maehara

j/w Tim Campion, Brandon Doherty, Chris Kapulkin

Institute of Mathematics for Industry, Kyushu University

ALGI 2021

Globular ω-categories

	Shape	Compositions
0-cells	$\bullet$	none
1-cells	$\bullet \longrightarrow \bullet$	$\bullet \longrightarrow \bullet \longrightarrow \bullet$

Globular ω-categories

	Shape	Compositions
0-cells	$\bullet$	none
1-cells	$\bullet \longrightarrow \bullet$	$\bullet \longrightarrow \bullet \longrightarrow \bullet$
2-cells	$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$	hor. $\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$ vert. $\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2ex, "\Downarrow"] \arrow[r, phantom, shift right=2ex, "\Downarrow"] & \bullet \end{tikzcd}$

	Shape	Compositions
0-cells	$\bullet$	none
1-cells	$\bullet \longrightarrow \bullet$	$\bullet \longrightarrow \bullet \longrightarrow \bullet$
2-cells	$\bullet \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \bullet$	hor. $\bullet \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \bullet \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \bullet$ vert. $\bullet \mathrel{\substack{\longrightarrow \\ \Downarrow \\ \longrightarrow \\ \Downarrow \\ \longrightarrow}} \bullet$
$\vdots$	$\vdots$	$\vdots$
n-cells	n-dimensional globe	n ways

Motivations coming from...

Motivations coming from...

- category theory itself

Motivations coming from...

- category theory itself
- homotopy theory
  (Homotopy Hypothesis etc.)

Motivations coming from...

- category theory itself
- homotopy theory

  (Homotopy Hypothesis etc.)
- non-abelian cohomology

  (as coefficient objects; *The algebra of oriented simplexes*, Street)

Motivations coming from...

- category theory itself
- homotopy theory
  (Homotopy Hypothesis etc.)
- non-abelian cohomology
  (as coefficient objects; *The algebra of oriented simplexes*, Street)
- topological quantum field theory

Why ω-categories?

Motivations coming from...

- category theory itself
- homotopy theory
  (Homotopy Hypothesis etc.)
- non-abelian cohomology
  (as coefficient objects; *The algebra of oriented simplexes*, Street)
- topological quantum field theory
- higher-dimensional rewriting theory
  (*Higher dimensional word problems with applications to equational logic*, Burroni)

Motivations coming from...

- category theory itself
- homotopy theory
  (Homotopy Hypothesis etc.)
- non-abelian cohomology
  (as coefficient objects; *The algebra of oriented simplexes*, Street)
- topological quantum field theory
- higher-dimensional rewriting theory
  (*Higher dimensional word problems with applications to equational logic*, Burroni)
- modelling concurrency
  (*Modeling concurrency with geometry*, Pratt)

We want to deal with weak ω-categories

Making things weak

We want to deal with weak ω-categories so *e.g.*

$$(hg)f \simeq h(gf)$$

Making things weak

We want to deal with weak ω-categories and weak ω-functors so *e.g.*

$$(hg)f \simeq h(gf)$$

We want to deal with weak ω-categories and weak ω-functors so *e.g.*

$$(hg)f \simeq h(gf) \qquad \text{and} \qquad F(gf) \simeq (Fg)(Ff).$$

Making things weak

We want to deal with weak ω-categories and weak ω-functors so *e.g.*

$$(hg)f \simeq h(gf) \qquad \text{and} \qquad F(gf) \simeq (Fg)(Ff).$$

But then we must worry about coherence

Making things weak

We want to deal with weak ω-categories and weak ω-functors so *e.g.*

$$(hg)f \simeq h(gf) \qquad \text{and} \qquad F(gf) \simeq (Fg)(Ff).$$

But then we must worry about coherence, *e.g.*

$$\begin{tikzcd}
 & & (kh)(gf) \arrow[drr, Rightarrow, "\alpha_{k,h,gf}"] & & \\
((kh)g)f \arrow[urr, Rightarrow, "\alpha_{kh,g,f}"] \arrow[dr, Rightarrow, "\alpha_{k,h,g}f"'] \arrow[rrrr, phantom, "\cong\Downarrow"] & & & & k(h(gf)) \\
 & (k(hg))f \arrow[rr, Rightarrow, "\alpha_{k,hg,f}"'] & & k((hg)f) \arrow[ur, Rightarrow, "k\alpha_{h,g,f}"'] &
\end{tikzcd}$$

Making things weak

We want to deal with weak ω-categories and weak ω-functors so *e.g.*

$$(hg)f \simeq h(gf) \qquad \text{and} \qquad F(gf) \simeq (Fg)(Ff).$$

But then we must worry about coherence, *e.g.*

$$\begin{tikzcd}
 & & (kh)(gf) \arrow[drr, Rightarrow, "\alpha_{k,h,gf}"] \arrow[dd, phantom, "\cong\Downarrow"] & & \\
((kh)g)f \arrow[urr, Rightarrow, "\alpha_{kh,g,f}"] \arrow[dr, Rightarrow, "\alpha_{k,h,g}f"'] & & & & k(h(gf)) \\
 & (k(hg))f \arrow[rr, Rightarrow, "\alpha_{k,hg,f}"'] & & k((hg)f) \arrow[ur, Rightarrow, "k\alpha_{h,g,f}"'] &
\end{tikzcd}$$

Idea

Define structures that can only "see" equivalences, and not equalities, between cells.

Making things weak

We want to deal with weak ω-categories and weak ω-functors so *e.g.*

$$(hg)f \simeq h(gf) \qquad \text{and} \qquad F(gf) \simeq (Fg)(Ff).$$

But then we must worry about coherence, *e.g.*

$$\begin{tikzcd}
& & (kh)(gf) \arrow[rrd, Rightarrow, "\alpha_{k,h,gf}"] \arrow[dd, phantom, "\cong\Downarrow"] & & \\
((kh)g)f \arrow[rru, Rightarrow, "\alpha_{kh,g,f}"] \arrow[rd, Rightarrow, "\alpha_{k,h,g}f"'] & & & & k(h(gf)) \\
& (k(hg))f \arrow[rr, Rightarrow, "\alpha_{k,hg,f}"'] & & k((hg)f) \arrow[ru, Rightarrow, "k\alpha_{h,g,f}"'] &
\end{tikzcd}$$

Idea

Define structures that can only "see" equivalences, and not equalities, between cells.

To realise this, simplicial, cubical, etc. are more convenient than globular.

	Shape	Compositions
0-cells	$\bullet$	none
1-cells	$\bullet \longrightarrow \bullet$	$\bullet \longrightarrow \bullet \longrightarrow \bullet$

	Shape	Compositions
0-cells	$\bullet$	none
1-cells	$\bullet \longrightarrow \bullet$	$\bullet \longrightarrow \bullet \longrightarrow \bullet$
2-cells	$\begin{tikzcd} & \bullet \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\ \bullet \arrow[ur] \arrow[rr] & {} & \bullet \end{tikzcd}$	V_1 $\begin{tikzcd} & & \bullet \arrow[ddr] \arrow[d, phantom, "\Uparrow"] & \\ & \bullet \arrow[ur] \arrow[drr] \arrow[d, phantom, "\Uparrow"] & {} & \\ \bullet \arrow[ur] \arrow[rrr] & {} & & \bullet \end{tikzcd}$ V_2 $\begin{tikzcd} & \bullet \arrow[dr] \arrow[d, phantom, "\Uparrow"] & & \\ & {} & \bullet \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\ \bullet \arrow[uur] \arrow[urr] \arrow[rrr] & & {} & \bullet \end{tikzcd}$

Simplicial ω-categories

	Shape	Compositions
0-cells	$\bullet$	none
1-cells	$\bullet \longrightarrow \bullet$	$\bullet \longrightarrow \bullet \longrightarrow \bullet$
2-cells	$\begin{tikzcd} & \bullet \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\ \bullet \arrow[ur] \arrow[rr] & {} & \bullet \end{tikzcd}$	V_1 $\begin{tikzcd} & & \bullet \arrow[ddr] & \\ & \bullet \arrow[ur] \arrow[drr] \arrow[ur, phantom, shift right=3, "\Uparrow"] & & \\ \bullet \arrow[ur] \arrow[rrr] \arrow[ur, phantom, shift right=3, "\Uparrow"] & & & \bullet \end{tikzcd}$ V_2 $\begin{tikzcd} & \bullet \arrow[dr] & & \\ & & \bullet \arrow[dr] & \\ \bullet \arrow[uur] \arrow[urr] \arrow[rrr] & & & \bullet \end{tikzcd}$
$\vdots$	$\vdots$	$\vdots$
n-cells	n-dimensional simplex	n ways

	Shape	Compositions
0-cells	$\bullet$	none
1-cells	$\bullet \longrightarrow \bullet$	$\bullet \longrightarrow \bullet \longrightarrow \bullet$
2-cells	$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[ur, Rightarrow] & \bullet \end{tikzcd}$	hor. $\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[ur, Rightarrow] & \bullet \arrow[r] \arrow[ur, Rightarrow] & \bullet \end{tikzcd}$ vert. $\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[d] \arrow[ur, Rightarrow] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[ur, Rightarrow] & \bullet \end{tikzcd}$
$\vdots$	$\vdots$	$\vdots$
n-cells	n-dimensional cube	n ways

Q1: But aren't they different things?

Q1: But aren't they different things?

Theorem (Al-Agl-Brown-Steiner, Steiner, Verity)

$$\begin{aligned}\{\textit{globular strict } \omega\textit{-categories}\} &\simeq \{\textit{simplicial strict } \omega\textit{-categories}\} \\ &\simeq \{\textit{cubical strict } \omega\textit{-categories}\}\end{aligned}$$

Q1: But aren't they different things?

Theorem (Al-Agl-Brown-Steiner, Steiner,Verity)

$$\{\textit{globular strict } \omega\textit{-categories}\} \simeq \{\textit{simplicial strict } \omega\textit{-categories}\} \simeq \{\textit{cubical strict } \omega\textit{-categories}\}$$

Q2: Why are weak things easier to deal with simplicially/cubically?

Q1: But aren't they different things?

Theorem (Al-Agl-Brown-Steiner, Steiner, Verity)

$$\begin{aligned}\{\textit{globular strict } \omega\textit{-categories}\} &\simeq \{\textit{simplicial strict } \omega\textit{-categories}\} \\ &\simeq \{\textit{cubical strict } \omega\textit{-categories}\}\end{aligned}$$

Q2: Why are weak things easier to deal with simplicially/cubically?

A2: We can encode compositions of n-cells using identity $(n+1)$-cells,

Simplicial weak ω-categories

Q1: But aren't they different things?

Theorem (Al-Agl-Brown-Steiner, Steiner, Verity)

$$\begin{aligned}\{\textit{globular strict } \omega\textit{-categories}\} &\simeq \{\textit{simplicial strict } \omega\textit{-categories}\} \\ &\simeq \{\textit{cubical strict } \omega\textit{-categories}\}\end{aligned}$$

Q2: Why are weak things easier to deal with simplicially/cubically?

A2: We can encode compositions of n-cells using identity $(n+1)$-cells,

e.g. any composable pair f, g of 1-cells

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] & & \bullet \end{tikzcd}$$

Simplicial weak ω-categories

Q1: But aren't they different things?

Theorem (Al-Agl-Brown-Steiner, Steiner,Verity)

$$\{\textit{globular strict } \omega\textit{-categories}\} \simeq \{\textit{simplicial strict } \omega\textit{-categories}\}$$
$$\simeq \{\textit{cubical strict } \omega\textit{-categories}\}$$

Q2: Why are weak things easier to deal with simplicially/cubically?

A2: We can encode compositions of n-cells using identity $(n+1)$-cells,

e.g. any composable pair f, g of 1-cells can be completed to

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] \arrow[d, phantom, "||"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, dashed] & {} & \bullet \end{tikzcd}$$

Simplicial weak ω-categories

Q1: But aren't they different things?

Theorem (Al-Agl-Brown-Steiner, Steiner, Verity)

$$
\begin{aligned}
\{\textit{globular strict } \omega\textit{-categories}\} &\simeq \{\textit{simplicial strict } \omega\textit{-categories}\} \\
&\simeq \{\textit{cubical strict } \omega\textit{-categories}\}
\end{aligned}
$$

Q2: Why are weak things easier to deal with simplicially/cubically?

A2: We can encode compositions of n-cells using identity $(n+1)$-cells, and "replace" them by suitable equivalences.

e.g. any composable pair f, g of 1-cells can be completed to

$$
\begin{tikzcd}
& \bullet \arrow[dr, "g"] & \\
\bullet \arrow[ur, "f"] \arrow[rr, dashed] & {} \arrow[u, phantom, "\wr|"] & \bullet
\end{tikzcd}
$$

Recall

The join of $\mathcal{C}, \mathcal{D} \in \underline{\mathrm{Cat}}$ is $\mathcal{C} \star \mathcal{D}$

Recall

The join of $\mathcal{C}, \mathcal{D} \in \underline{\mathrm{Cat}}$ is $\mathcal{C} \star \mathcal{D}$ given by $\mathrm{ob}(\mathcal{C} \star \mathcal{D}) = \mathrm{ob}\,\mathcal{C} \amalg \mathrm{ob}\,\mathcal{D}$ and

Recall

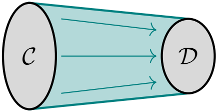
The join of $\mathcal{C}, \mathcal{D} \in \underline{\mathrm{Cat}}$ is $\mathcal{C} \star \mathcal{D}$ given by $\mathrm{ob}(\mathcal{C} \star \mathcal{D}) = \mathrm{ob}\,\mathcal{C} \amalg \mathrm{ob}\,\mathcal{D}$ and

$$(\mathcal{C} \star \mathcal{D})(x, y) = \begin{cases} \mathcal{C}(x, y), & x, y \in \mathcal{C} \\ \mathcal{D}(x, y), & x, y \in \mathcal{D} \\ \{*\}, & x \in \mathcal{C}, y \in \mathcal{D} \\ \varnothing, & x \in \mathcal{D}, y \in \mathcal{C} \end{cases}$$

Recall

The join of $\mathcal{C}, \mathcal{D} \in \underline{\mathrm{Cat}}$ is $\mathcal{C} \star \mathcal{D}$ given by $\mathrm{ob}(\mathcal{C} \star \mathcal{D}) = \mathrm{ob}\,\mathcal{C} \amalg \mathrm{ob}\,\mathcal{D}$ and

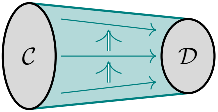
$$(\mathcal{C} \star \mathcal{D})(x,y) = \begin{cases} \mathcal{C}(x,y), & x,y \in \mathcal{C} \\ \mathcal{D}(x,y), & x,y \in \mathcal{D} \\ \{*\}, & x \in \mathcal{C}, y \in \mathcal{D} \\ \varnothing, & x \in \mathcal{D}, y \in \mathcal{C} \end{cases}$$



Recall

The join of $\mathcal{C}, \mathcal{D} \in \underline{\mathrm{Cat}}$ is $\mathcal{C} \star \mathcal{D}$ given by $\mathrm{ob}(\mathcal{C} \star \mathcal{D}) = \mathrm{ob}\,\mathcal{C} \amalg \mathrm{ob}\,\mathcal{D}$ and

$$(\mathcal{C} \star \mathcal{D})(x, y) = \begin{cases} \mathcal{C}(x, y), & x, y \in \mathcal{C} \\ \mathcal{D}(x, y), & x, y \in \mathcal{D} \\ \{*\}, & x \in \mathcal{C}, y \in \mathcal{D} \\ \varnothing, & x \in \mathcal{D}, y \in \mathcal{C} \end{cases}$$



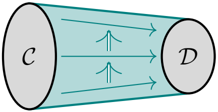
join of ω-categories

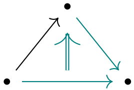
Join

Recall

The join of $\mathcal{C}, \mathcal{D} \in \underline{\mathrm{Cat}}$ is $\mathcal{C} \star \mathcal{D}$ given by $\mathrm{ob}(\mathcal{C} \star \mathcal{D}) = \mathrm{ob}\,\mathcal{C} \amalg \mathrm{ob}\,\mathcal{D}$ and

$$(\mathcal{C} \star \mathcal{D})(x, y) = \begin{cases} \mathcal{C}(x, y), & x, y \in \mathcal{C} \\ \mathcal{D}(x, y), & x, y \in \mathcal{D} \\ \{*\}, & x \in \mathcal{C}, y \in \mathcal{D} \\ \varnothing, & x \in \mathcal{D}, y \in \mathcal{C} \end{cases}$$





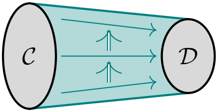
join of ω-categories

Join

Recall

The join of $\mathcal{C}, \mathcal{D} \in \underline{\mathrm{Cat}}$ is $\mathcal{C} \star \mathcal{D}$ given by $\mathrm{ob}(\mathcal{C} \star \mathcal{D}) = \mathrm{ob}\,\mathcal{C} \amalg \mathrm{ob}\,\mathcal{D}$ and

$$(\mathcal{C} \star \mathcal{D})(x, y) = \begin{cases} \mathcal{C}(x, y), & x, y \in \mathcal{C} \\ \mathcal{D}(x, y), & x, y \in \mathcal{D} \\ \{*\}, & x \in \mathcal{C}, y \in \mathcal{D} \\ \varnothing, & x \in \mathcal{D}, y \in \mathcal{C} \end{cases}$$



$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & \arrow[u, phantom, "\Uparrow"] & \bullet \end{tikzcd}$$

join of ω-categories

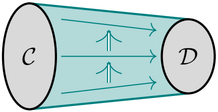
$$[m] \star [n] = [m + n + 1].$$

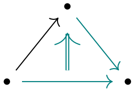
Join

Recall

The join of $\mathcal{C}, \mathcal{D} \in \underline{\mathrm{Cat}}$ is $\mathcal{C} \star \mathcal{D}$ given by $\mathrm{ob}(\mathcal{C} \star \mathcal{D}) = \mathrm{ob}\,\mathcal{C} \amalg \mathrm{ob}\,\mathcal{D}$ and

$$(\mathcal{C} \star \mathcal{D})(x, y) = \begin{cases} \mathcal{C}(x, y), & x, y \in \mathcal{C} \\ \mathcal{D}(x, y), & x, y \in \mathcal{D} \\ \{*\}, & x \in \mathcal{C}, y \in \mathcal{D} \\ \varnothing, & x \in \mathcal{D}, y \in \mathcal{C} \end{cases}$$





Can model join of ω-categories using join of simplicial sets which extends

$$[m] \star [n] = [m + n + 1].$$

Recall

The Gray tensor product on $2\text{-}\underline{\mathrm{Cat}}$

Recall

The Gray tensor product on $2\text{-}\underline{\mathrm{Cat}}$ is a tensor product such that

$$\left.\begin{array}{l} x \xrightarrow{\quad f \quad} x' \text{ in } \mathcal{C} \\[2em] y \xrightarrow{\quad g \quad} y' \text{ in } \mathcal{D} \end{array}\right\} \quad \rightsquigarrow \quad \begin{tikzcd} (x,y) \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] & (x',y) \arrow[d, "{(x',g)}"] \\ (x,y') \arrow[r, "{(f,y')}"'] & (x',y') \end{tikzcd}$$

Recall

The Gray tensor product on $2\text{-}\underline{\mathrm{Cat}}$ is a tensor product such that

$$\left.\begin{array}{l} x \xrightarrow{\quad f \quad} x' \text{ in } \mathcal{C} \\[2em] y \xrightarrow{\quad g \quad} y' \text{ in } \mathcal{D} \end{array}\right\} \quad \rightsquigarrow \quad \begin{tikzcd} (x,y) \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] \arrow[dr, phantom, "\nearrow"] & (x',y) \arrow[d, "{(x',g)}"] \\ (x,y') \arrow[r, "{(f,y')}"'] & (x',y') \end{tikzcd}$$

Recall

The Gray tensor product on $2\text{-}\underline{\mathrm{Cat}}$ is a tensor product such that

$$\left.\begin{array}{l} x \xrightarrow{\quad f \quad} x' \text{ in } \mathcal{C} \\[2em] y \xrightarrow{\quad g \quad} y' \text{ in } \mathcal{D} \end{array}\right\} \quad \rightsquigarrow \quad \begin{tikzcd} (x,y) \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] \arrow[dr, phantom, "\nearrow"] & (x',y) \arrow[d, "{(x',g)}"] \\ (x,y') \arrow[r, "{(f,y')}"'] & (x',y') \end{tikzcd}$$

subject to

$$\left\{\begin{array}{c} \begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, "\mathrm{id}"'] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "/\!/"] & \bullet \arrow[d] \\ \bullet \arrow[r, "\mathrm{id}"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend right] & \bullet \end{tikzcd} \\ \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[rr] \arrow[d] \arrow[drr, phantom, "\nearrow"] & & \bullet \arrow[d] \\ \bullet \arrow[rr] & & \bullet \end{tikzcd} \end{array}\right\} \text{ etc.}$$

Recall

The Gray tensor product on $2\text{-}\underline{\mathrm{Cat}}$ is a tensor product such that

$$\left.\begin{array}{l} x \xrightarrow{\quad f \quad} x' \text{ in } \mathcal{C} \\[2em] y \xrightarrow{\quad g \quad} y' \text{ in } \mathcal{D} \end{array}\right\} \quad \rightsquigarrow \quad \begin{tikzcd} (x,y) \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] \arrow[dr, phantom, "\Nearrow"] & (x',y) \arrow[d, "{(x',g)}"] \\ (x,y') \arrow[r, "{(f,y')}"'] & (x',y') \end{tikzcd}$$

subject to

$$\left\{ \begin{array}{c}
\begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, "\mathrm{id}"'] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "/\!/"] & \bullet \arrow[d] \\ \bullet \arrow[r, "\mathrm{id}"'] & \bullet \end{tikzcd}
\qquad
\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend right] & \bullet \end{tikzcd}
\\
\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Nearrow"] & & \bullet \arrow[d] \\ \bullet \arrow[rr] & & \bullet \end{tikzcd}
\end{array} \right\} \quad \text{etc.}$$

Gray tensor product of ω-categories

Recall

The Gray tensor product on $2\text{-}\underline{\mathrm{Cat}}$ is a tensor product such that

$$\left.\begin{array}{l} x \xrightarrow{\quad f \quad} x' \text{ in } \mathcal{C} \\[2em] y \xrightarrow{\quad g \quad} y' \text{ in } \mathcal{D} \end{array}\right\} \quad \rightsquigarrow \quad \begin{tikzcd} (x,y) \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] \arrow[dr, phantom, "\nearrow"] & (x',y) \arrow[d, "{(x',g)}"] \\ (x,y') \arrow[r, "{(f,y')}"'] & (x',y') \end{tikzcd}$$

subject to

$$\left\{\begin{array}{c}
\begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, "\mathrm{id}"'] & \bullet \end{tikzcd}
=
\begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "/\!/"] & \bullet \arrow[d] \\ \bullet \arrow[r, "\mathrm{id}"'] & \bullet \end{tikzcd}
\qquad
\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \bullet \end{tikzcd}
=
\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend right] & \bullet \end{tikzcd}
\\
\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}
=
\begin{tikzcd} \bullet \arrow[rr] \arrow[d] \arrow[drr, phantom, "\nearrow"] & & \bullet \arrow[d] \\ \bullet \arrow[rr] & & \bullet \end{tikzcd}
\end{array}\right\} \quad \text{etc.}$$

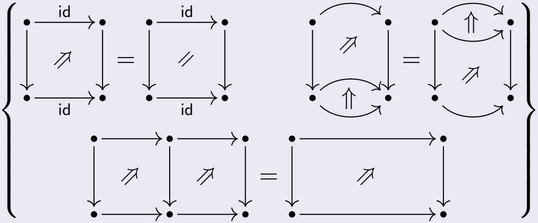
Gray tensor product of ω-categories

$$m\text{-cube} \otimes n\text{-cube} = (m+n)\text{-cube}.$$

Recall

The Gray tensor product on 2-$\underline{\mathrm{Cat}}$ is a tensor product such that

$$\left.\begin{array}{l} x \xrightarrow{\ f\ } x' \text{ in } \mathcal{C} \\[2em] y \xrightarrow{\ g\ } y' \text{ in } \mathcal{D} \end{array}\right\} \quad \rightsquigarrow \quad \begin{tikzcd} (x,y) \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] \arrow[dr, phantom, "\Nearrow"] & (x',y) \arrow[d, "{(x',g)}"] \\ (x,y') \arrow[r, "{(f,y')}"'] & (x',y') \end{tikzcd}$$

subject to 

etc.

Can model Gray tensor product of ω-categories using geometric product of cubical sets which extends

$$m\text{-cube} \otimes n\text{-cube} = (m+n)\text{-cube}.$$

Recall

The Gray tensor product on $2\text{-}\underline{\mathrm{Cat}}$ is a tensor product such that

$$\left.\begin{array}{l} x \xrightarrow{\quad f \quad} x' \text{ in } \mathcal{C} \\[3em] y \xrightarrow{\quad g \quad} y' \text{ in } \mathcal{D} \end{array}\right\} \quad \rightsquigarrow \quad \begin{tikzcd} (x,y) \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] \arrow[dr, phantom, "\Nearrow"] & (x',y) \arrow[d, "{(x',g)}"] \\ (x,y') \arrow[r, "{(f,y')}"'] & (x',y') \end{tikzcd}$$

subject to

$$\left\{ \begin{array}{c} \begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, "\mathrm{id}"'] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "/\!/"] & \bullet \arrow[d] \\ \bullet \arrow[r, "\mathrm{id}"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r, bend right] & \bullet \end{tikzcd} \\ \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Nearrow"] & & \bullet \arrow[d] \\ \bullet \arrow[rr] & & \bullet \end{tikzcd} \end{array} \right\} \quad \text{etc.}$$

Can model (both lax and pseudo) Gray tensor product of ω-categories using geometric product of cubical sets which extends

$$m\text{-cube} \otimes n\text{-cube} = (m+n)\text{-cube}.$$

Main result

Theorem (Campion-Kapulkin-M., Doherty-Kapulkin-M.)

There is a Quillen equivalence between:

Main result

Theorem (Campion-Kapulkin-M., Doherty-Kapulkin-M.)

There is a Quillen equivalence between:

- *Verity's complicial model structure on marked simplicial sets; and*

Main result

Theorem (Campion-Kapulkin-M., Doherty-Kapulkin-M.)

There is a Quillen equivalence between:

- *Verity's complicial model structure on marked simplicial sets; and*
- *comical model structure on marked cubical sets.*

Main result

Theorem (Campion-Kapulkin-M., Doherty-Kapulkin-M.)

There is a Quillen equivalence between:

- *Verity's complicial model structure on marked simplicial sets; and*
- *comical model structure on marked cubical sets.*

Moreover this equivalence preserves (both lax and pseudo) Gray tensor product.

Theorem (Campion-Kapulkin-M., Doherty-Kapulkin-M.)

There is a Quillen equivalence between:

- *Verity's complicial model structure on marked simplicial sets; and*
- *comical model structure on marked cubical sets.*

Moreover this equivalence preserves (both lax and pseudo) Gray tensor product.

In other words, the weak ω-categories of

- simplicial weak ω-categories (complicial sets); and
- cubical weak ω-categories (comical sets)

are suitably equivalent.

Main result

Theorem (Campion-Kapulkin-M., Doherty-Kapulkin-M.)

There is a Quillen equivalence between:

- *Verity's complicial model structure on marked simplicial sets; and*
- *comical model structure on marked cubical sets.*

Moreover this equivalence preserves (both lax and pseudo) Gray tensor product.

In other words, the weak ω-categories of

- simplicial weak ω-categories (complicial sets); and
- cubical weak ω-categories (comical sets)

are suitably equivalent.

Preprints available at:

- [CKM] A cubical model for (∞, n)-categories (arXiv:2005.07603)
- [DKM] Equivalence of cubical and simplicial approaches to (∞, n)-categories (arXiv:2106.09428)