AUGMENTED SIMPLICIAL SETS AS M-SETS

YUKI MAEHARA

Abstract. We show that the category of augmented simplicial sets is comonadic over the category of M-sets, where M is the monoid of *eventually consecutive* endomorphisms on the ordinal ω.

1. Introduction

The purpose of this short note is exhibit the category $\underline{\mathrm{sSet}}_+ = [\Delta_+^{\mathrm{op}}, \underline{\mathrm{Set}}]$ of *augmented simplicial sets* as being comonadic over the category of M-sets for a suitable monoid M.

The intuition behind this result is as follows. The category Δ_+ consists of the standard n-simplices for various n. For any $m \leq n$, the m-simplex $[m]$ may be realised as an initial segment of the n-simplex $[n]$. Thus if we have access to the "standard ω-simplex", then we may expect the monoid of endomorphisms on this object to subsume the category Δ_+ in some sense. Indeed, the monoid M will be defined as comprising of certain endomorphisms on the ordinal $\omega = \{0 < 1 < \dots\}$.

The adjunction between $\underline{\mathrm{sSet}}_+$ and M-$\underline{\mathrm{Set}}$ described below may be informally thought of as follows. The left adjoint $U : \underline{\mathrm{sSet}}_+ \to M\text{-}\underline{\mathrm{Set}}$ "pads out" all (finite-dimensional) simplices in a given augmented simplicial set to ω-dimensional simplices. The right adjoint $R : M\text{-}\underline{\mathrm{Set}} \to \underline{\mathrm{sSet}}_+$ sends each M-set X to the augmented simplicial set RX whose n-simplices are precisely those $x \in X$ whose "tail" past n is a "padding". A coalgebra structure for the induced comonad then corresponds to assigning a finite "actual" dimension to each ω-simplex in a coherent manner.

2. Augmented simplicial sets

Let Δ_+ denote the category of finite ordinals

$$[n] = \{0 < \cdots < n\}$$

and order-preserving maps. It is well known that this category may be presented by the generating morphisms

$$\delta_k^n : [n-1] \to [n], \quad \sigma_k^n : [n+1] \to [n]$$

for $0 \leq k \leq n$ subject to the cosimplicial identities.

We will write $\underline{\mathrm{sSet}}_+$ for the category $[\Delta_+^{\mathrm{op}}, \underline{\mathrm{Set}}]$ of *augmented simplicial sets.* For any $A \in \underline{\mathrm{sSet}}_+$, $a \in A_n \overset{\mathrm{def}}{=} A\big([n]\big)$ and $\alpha : [m] \to [n]$ in Δ_+, we write $a \cdot \alpha$ to mean $A(\alpha)(a) \in A_m$.

3. The monoid M

We will write ω for the ordinal $\omega = \{0 < 1 < \dots\}$.

Definition 3.1. An order-preserving function $\mu : \omega \to \omega$ is called *eventually consecutive* if there exists $n \in \omega$ such that $\mu(i) = \mu(n) + (i - n)$ for all $i \geq n$.

Example. For $k \in \mathbb{N}$, let $\delta_k, \sigma_k : \omega \to \omega$ be the functions given by

$$\delta_k(i) = \begin{cases} i, & i < k, \\ i+1, & i \geq k \end{cases}$$

and

$$\sigma_k(i) = \begin{cases} i, & i \leq k, \\ i-1, & i > k. \end{cases}$$

These functions are eventually consecutive for any k.

Definition 3.2. Let M denote the monoid (under composition) of all eventually consecutive, order-preserving functions $\mu : \omega \to \omega$.

Definition/Warning 3.3. We will write "max" for the function that takes a finite initial segment $[n]$ of ω and returns the value n. In particular, $\max \varnothing = -1$.

Definition 3.4. For any $\mu \in M$ and $n \in \omega$, we denote by $\mu|_n$ the restriction

$$\mu|_n : [m] \to [n]$$

of μ where $m = \max\bigl(\mu^{-1}[n]\bigr)$. We write $\mu \sim_n \nu$ if $\mu|_n = \nu|_n$ holds for $\mu, \nu \in M$.

The following propositions are straightforward to verify.

Proposition 3.5. *The equality $(\mu\nu)|_n = \mu|_n \nu|_m$ holds for any $\mu, \nu \in M$ and $n \in \omega$ where $m = \max\bigl(\mu^{-1}[n]\bigr)$.*

Proposition 3.6. *For each $n \in \omega$, the binary relation $\sim_n$ on M is an equivalence relation. Moreover, $\mu \sim_n \mu'$ implies $\mu\nu \sim_n \mu'\nu$ for any $\mu, \mu', \nu \in M$.*

Definition 3.7. We will denote the $\sim_n$-class containing $\mu \in M$ by $[\mu]_n$.

4. Adjunction between $\underline{\mathrm{sSet}}_+$ and M-$\underline{\mathrm{Set}}$

Proposition 3.6 implies that, for each $n \in \omega$, the quotient $M/{\sim_n}$ admits a right M-action given by $[\mu]_n \cdot \nu = [\mu\nu]_n$. We will extend the assignation $n \mapsto M/{\sim_n}$ to a functor $\Delta_+ \to M\text{-}\underline{\mathrm{Set}} \overset{\text{def}}{=} [M^{\mathrm{op}}, \underline{\mathrm{Set}}]$ (where M is regarded as a one-object category).

Definition 4.1. We denote by $\overrightarrow{(-)} : \Delta_+ \to M$ the functor that sends each simplicial operator $\alpha : [m] \to [n]$ to the function $\overrightarrow{\alpha} : \omega \to \omega$ given by

$$\overrightarrow{\alpha}(i) = \begin{cases} \alpha(i), & i \leq m, \\ n + (i - m), & i > m. \end{cases}$$

Example. We have $\overrightarrow{\delta^n_k} = \delta_k$ and $\overrightarrow{\sigma^n_k} = \sigma_k$ for any $0 \leq k \leq n$.

It is easy to check that the operations $(-)|_n$ and $\overrightarrow{(-)}$ behave like mutual inverses in the following sense.

Proposition 4.2. *For any $\alpha : [m] \to [n]$ in Δ_+, we have $\overrightarrow{\alpha}|_n = \alpha$. For any $\mu \in M$ and $n \in \omega$, we have $\overrightarrow{\mu|_n} \sim_n \mu$; equivalently, $\left[\overrightarrow{\mu|_n}\right]_n = [\mu]_n$.*

Proposition 4.3. *For any $\alpha : [m] \to [n]$ in Δ_+, post-composing with $\overrightarrow{\alpha} : \omega \to \omega$ induces a morphism $M/{\sim_m} \to M/{\sim_n}$ in M-$\underline{\mathrm{Set}}$. Moreover, this assignation defines a functor $F : \Delta_+ \to M$-$\underline{\mathrm{Set}}$.*

Proof. Let $\mu, \nu \in M$ and suppose $\mu|_m = \nu|_m$. Then we have

$$(\overrightarrow{\alpha}\mu)|_n = \overrightarrow{\alpha}|_n \mu|_m = \overrightarrow{\alpha}|_n \nu|_m = (\overrightarrow{\alpha}\nu)|_n$$

by Proposition 3.5. This shows that $\overrightarrow{\alpha}$ induces a function $M/{\sim_m} \to M/{\sim_n}$. The rest of the statement is straightforward to check. □

The functor F induces an adjunction via the Kan construction. We will give an explicit description of the resulting adjunction $U \dashv R$.

Definition 4.4. For any $A \in \underline{\mathrm{sSet}}_+$, let $UA \in M\text{-}\underline{\mathrm{Set}}$ be the coproduct $\coprod_{m \geq -1} A_m$ equipped with the right M-action $a \cdot \mu \overset{\text{def}}{=} a \cdot \mu|_m$ for $a \in A_m$. This extends to a functor $U : \underline{\mathrm{sSet}}_+ \to M\text{-}\underline{\mathrm{Set}}$ in the obvious way.

Remark. That UA is indeed an M-set follows from Proposition 3.5 and the fact that $\mathrm{id}_\omega|_m = \mathrm{id}_{[m]}$ holds for any $m \geq -1$.

Proposition 4.5. *The functor U is the left Kan extension of F along the Yoneda embedding.*

Note that the square

$$\begin{tikzcd} \underline{\mathrm{sSet}}_+ \arrow[r, "U"] \arrow[d, "V"'] & M\text{-}\underline{\mathrm{Set}} \arrow[d, "W"] \\ \underline{\mathrm{Set}}^{\mathrm{ob}\,\Delta_+} \arrow[r, "\coprod"'] & \underline{\mathrm{Set}} \end{tikzcd} \tag{1}$$

commutes by our definition of U, where V and W are induced by the discrete inclusions $\mathrm{ob}\,\Delta_+ \hookrightarrow \Delta_+$ and $\mathrm{ob}\,M \hookrightarrow M$ respectively, and $\coprod$ is the coproduct functor.

Proof. Since V and $\coprod$ preserve colimits and W reflects them, it follows from the commutativity of (1) that U preserves them. So it suffices to exhibit a natural isomorphism $U\big(\Delta_+^n\big) \cong F[n]$.

Fix $n \geq -1$, and let

$$\iota_n : \coprod_{m \geq -1} \Delta_+\big([m],[n]\big) \to M/{\sim_n}$$

be the function given by sending $\alpha : [m] \to [n]$ to $[\overrightarrow{\alpha}]_n$. Then for any $\mu \in M$, we have

$$\begin{aligned}
\iota_n(\alpha) \cdot \mu &= [\overrightarrow{\alpha}]_n \cdot \mu && \text{definition of } \iota_n \\
&= [\overrightarrow{\alpha}\mu]_n && M\text{-action on } M/{\sim_n} \\
&= \Big[\overrightarrow{(\overrightarrow{\alpha}\mu)|_n}\Big]_n && \text{Proposition 4.2} \\
&= \Big[\overrightarrow{\overrightarrow{\alpha}|_n \mu|_m}\Big]_n && \text{Proposition 3.5} \\
&= \Big[\overrightarrow{\alpha(\mu|_m)}\Big]_n && \text{Proposition 4.2} \\
&= \big[\overrightarrow{\alpha \cdot \mu}\big]_n && M\text{-action on } U\big(\Delta_+^n\big) \\
&= \iota_n(\alpha \cdot \mu) && \text{definition of } \iota_n.
\end{aligned}$$

Thus ι_n is a morphism in M-$\underline{\mathrm{Set}}$.

This map ι_n is a surjection since $[\mu]_n = \left[\overrightarrow{\mu|_n}\right]_n = \iota_n(\mu|_n)$ holds for any $\mu \in M$ by Proposition 4.2.

To see that ι_n is an injection, let $\alpha : [m] \to [n]$ in Δ_+. Observe that $\alpha = \mu|_n$ holds for any $\mu \in M$ with $\overrightarrow{\alpha} \sim_n \mu$. Thus α may be recovered from $\iota_n(\alpha) = [\overrightarrow{\alpha}]_n$ by choosing any representative and applying $(-)|_n$.

The naturality of ι_n in $[n]$ is easy to check, and this completes the proof. $\square$

Definition 4.6. Let $X \in M\text{-}\underline{\mathrm{Set}}$, $x \in X$ and $n \in \omega$. We write $\dim(x) \leq n$ to mean that $x \cdot \mu = x \cdot \nu$ holds for any $\mu, \nu \in M$ with $\mu \sim_n \nu$.

Definition 4.7. For any $X \in M\text{-}\underline{\mathrm{Set}}$, let RX denote the augmented simplicial set given by

$$(RX)_n = \big\{(x, n) : x \in X,\ \dim(x) \leq n\big\}$$

for $n \geq -1$ and $(x, n) \cdot \alpha = (x \cdot \overrightarrow{\alpha}, m)$ for $\alpha : [m] \to [n]$ in Δ_+. This extends to a functor $R : M\text{-}\underline{\mathrm{Set}} \to \underline{\mathrm{sSet}}_+$ in the obvious way.

Proposition 4.8. *There exist bijections*

$$M\text{-}\underline{\mathrm{Set}}(F[n], X) \cong (RX)_n$$

natural in $[n] \in \Delta_+$ *and* $X \in M\text{-}\underline{\mathrm{Set}}$.

Proof. Note that a map $f : F[n] = M/{\sim_m} \to X$ in $M\text{-}\underline{\mathrm{Set}}$ is completely determined by its value on $[\mathrm{id}]_n$. More precisely, if $f\big([\mathrm{id}]_n\big) = x$ then

$$\begin{aligned} f\big([\mu]_n\big) &= f\big([\mathrm{id}]_n \cdot \mu\big) && M\text{-action on } M/{\sim_n} \\ &= f\big([\mathrm{id}]_n\big) \cdot \mu && f \text{ respects } M\text{-action} \\ &= x \cdot \mu \end{aligned}$$

holds for any $\mu \in M$. It is clear from this formula that such an element $x \in X$ must satisfy $\dim(x) \leq n$. Conversely, if $x \in X$ satisfies $\dim(x) \leq n$ then the assignation $[\mu]_n \mapsto x \cdot \mu$ can be easily shown to be a well-defined morphism $M/{\sim_n} \to X$ in $M\text{-}\underline{\mathrm{Set}}$. This establishes the desired bijection, and its naturality is straightforward to check. $\square$

Corollary 4.9. *Applying the Kan construction to the functor* $F : \Delta_+ \to M\text{-}\underline{\mathrm{Set}}$ *yields the adjunction* $U \dashv R$.

5. Comonadicity

Now we prove the main theorem.

Theorem 5.1. *The left adjoint functor* $U : \underline{\mathrm{sSet}}_+ \to M\text{-}\underline{\mathrm{Set}}$ *is comonadic.*

Proof. We will use a version of Beck's monadicity theorem. Note that $\underline{\mathrm{sSet}}_+$ is complete and in particular has U-split equalisers.

Consider the commutative square (1). Since both V and $\coprod$ are conservative, so is U. Also, since V and $\coprod$ preserve connected limits and W reflects them, it follows that U preserves them. In particular, U preserves U-split equalisers. This completes the proof. $\square$

The induced comonad $\mathbb{G}$ on $M\text{-}\underline{\mathrm{Set}}$ may be discribed as follows. The functor part G sends each $X \in M\text{-}\underline{\mathrm{Set}}$ to

$$GX = \big\{(x, n) \in X \times \omega : \dim(x) \leq n\big\}$$

equipped with the M-action

$$(x, n) \cdot \mu = \big(x \cdot \mu, \max\big(\mu^{-1}[n]\big)\big).$$

The morphism $Gf : GX \to GY$ sends (x, n) to $(f(x), n)$. The counit $GX \to X$ is given by the first projection $(x, n) \mapsto x$, and the comultiplication $GX \to GGX$ is given by $(x, n) \mapsto \big((x, n), n\big)$.

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